The 71st William Lowell Putnam Mathematical Competition
Saturday, December 4, 2010

Done!

A1 Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is *at least* 3.]

Solution:

The number k is impossible unless $n(n+1)/2$, which is the sum of the n numbers, is divisible by k. Evidently we must also have $k < n$ for otherwise either a box would be empty or each would have one number in it which would not solve the problem. If n is even then we make take $k = n/2$ by putting 2 numbers in each box: $\{1, n\}, \{2, n-1\}, \cdots$. If $n = 2m+1$ is odd then we put n in a box and use the previous solution on the numbers from 1 to $2m = n-1$. This gives us $k = (n+1)/2$. If we try to use a larger k then the sum in each box must be smaller than n so there is no place to put the number n! The solution is $k = \lfloor (n+1)/2 \rfloor$ which is $n/2$ for n even and $(n+1)/2$ for n odd.

A2 Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

Solution:

Put $x = 0$ to discover that for all integers $n > 0$ we have

$$f(n) = f(0) + nf'(0)$$

which puts $f(n)$ on the line with slope $f'(0)$ and intercept $f(0)$. We have

$$f'(x+1) = \frac{f(x+(n+1)) - f(x) - (f(x+1) - f(x))}{n} = \frac{(n+1)f'(x)}{n} - \frac{f'(x)}{n} = f'(x).$$

So f' is periodic with period 1! Thus

$$\int_y^{y+1} f'(x)dx = \int_0^1 f'(x)dx$$

and

$$f(x+n) - f(x) = n\int_0^1 f'(u)du$$

is free of x. Thus f' is constant and f is a straight line. Every straight line function satisfies the condition so the collection of all such f is just the collection of all straight line functions $y = ax + b$.

A3 Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x,y) = a\frac{\partial h}{\partial x}(x,y) + b\frac{\partial h}{\partial y}(x,y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x,y)| \leq M$ for all $(x,y) \in \mathbb{R}^2$, then h is identically zero.

Solution:

Put

$$g(u) = h(au, bu + c)$$

and note that

$$\frac{dg}{du}(u) = a\frac{\partial h}{\partial x}(x, y)|_{x=au,y=bu+c} + b\frac{\partial h}{\partial y}(x, y)|_{x=au,y=bu+c} = h(au, bu + c) = g(u).$$

It follows that $g(u) = d\exp(u)$ *which is bounded over all* u *if and only if* $d = 0$*. Thus if* h *is bounded then* $h(au, bu + c) \equiv 0$ *for all choices of* c*. This evidently implies* h *is identically 0. If* $a = 0$ *or* $b = 0$ *use* $g(u) = h(x, bu)$ *or* $g(u) = h(au, b)$ *while for* $a = b = 0$ *we have* $h \equiv 0$ *is given.*

A4 Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

Solution:

Since 10 is congruent to -1 mod 11 we find that for even n

$$10^n \equiv 1 \mod 11$$

while if n *is odd then*

$$10^n \equiv -1 \mod 11$$

All powers of 10 greater than 1 are even so if n *is odd the sum given is congruent to 0 mod 11. Since the sum is not 11 or 0 it is not prime. Now suppose* $n = p2^m$ *where* $m > 0$ *and* p *is odd. We compute the residue class of the given number modulo* 10^{2^m}*. The last two terms are each congruent to -1. I claim the first two terms are congruent to 1 which would finish the problem. Each of those terms has the form* 10^r *and it suffices to show that the power* r *is divisible by* 2^{m+1} *for in that case*

$$10^r = (10^{2^m})^{2r/(2^{m+1})} \equiv ((-1)^2)^{r/2^{m+1}} \mod 10^{2^m} + 1.$$

In the case of the second term

$$r = 10^n = 10^{p2^m}$$

which is divisible by 2^{m+1} *provided* $m + 1 \le p2^m$ *which is true for all* $m \ge 0$ *and all odd positive integers* p*.*

A5 Let G be a group, with operation $*$. Suppose that

(i) G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);

(ii) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = 0$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

Prove that $\mathbf{a} \times \mathbf{b} = 0$ for all $\mathbf{a}, \mathbf{b} \in G$.

Solution:

First pick any $\mathbf{a}$ *and* $\mathbf{b}$ *both in* G *and not parallel (that is with non-zero cross product). (If no such pair exists we are done.) Then*

$$\mathbf{c} \equiv \mathbf{a} \times \mathbf{b} \ne 0$$

so $\mathbf{c} = \mathbf{a} * \mathbf{b}$ *is in the group. These three vectors are linearly independent. Now consider the identity* 1*. We cannot have* 1 *perpendicular to all three of the previous vectors unless* $1 = 0$*. If* 1 *is not 0 then its cross-product with at least one of the vectors is not 0 so suppose without loss that* $\mathbf{a} \times 1 \ne 0$*. Then* $\mathbf{a} \times 1 = \mathbf{a} * 1 = \mathbf{a}$ *which is not perpendicular to* $\mathbf{a}$ *— a contradiction. It follows that the group identity is the 0 vector. Now consider, for any non-zero* $\mathbf{x}$ *in* G*,*

$$\mathbf{x} * \mathbf{x}^{-1} = 1$$

where the inverse is the group inverse. Note that $\mathbf{x}^{-1} \ne 1 = 0$ *for otherwise we get* $\mathbf{x} = 1 = 0$*. We then have either*

$$\mathbf{x} \times \mathbf{x}^{-1} = \mathbf{x} * \mathbf{x}^{-1} = 1 = 0$$

or

$$\mathbf{x} \times \mathbf{x}^{-1} = 0$$

so either way

$$\mathbf{x} \times \mathbf{x}^{-1} = \mathbf{x} * \mathbf{x}^{-1} = 0.$$

It follows that $\mathbf{x}^{-1}$ *is parallel to* $\mathbf{x}$*. Now consider*

$$\begin{aligned} \mathbf{b} &= (\mathbf{a}^{-1} * \mathbf{a}) * \mathbf{b} \\ &= \mathbf{a}^{-1} * (\mathbf{a} * \mathbf{b}) \\ &= \mathbf{a}^{-1} * \mathbf{c} \\ &= \mathbf{a}^{-1} \times \mathbf{c}. \end{aligned}$$

We see that $\mathbf{b}$ *is perpendicular to* $\mathbf{c}$ *and to* $\mathbf{a}^{-1}$ *and so also to* $\mathbf{a}$*. Thus* $\mathbf{a}$*,* $\mathbf{b}$ *and* $\mathbf{c}$ *are mutually perpendicular. Now suppose* $\mathbf{x}$ *and* $\mathbf{y}$ *are two parallel non-zero vectors in* G*. I claim* $\mathbf{x} * \mathbf{y}$ *is parallel to* $\mathbf{x}$ *and* $\mathbf{y}$*. If not then*

$$\mathbf{y} = \mathbf{x}^{-1} * \mathbf{x} * \mathbf{y} = \mathbf{x}^{-1} \times \mathbf{x} * \mathbf{y}$$

is perpendicular to $\mathbf{x}^{-1}$ *which is parallel to* $\mathbf{y}$ *generating a contradiction. Now consider*

$$\mathbf{x} \equiv \mathbf{a} * \mathbf{b} * \mathbf{c}$$

On the one hand $\mathbf{a} * \mathbf{b}$ *is parallel to* $\mathbf{c}$ *so* $\mathbf{x}$ *is parallel to* $\mathbf{c}$*. On the other hand* $\mathbf{b} * \mathbf{c}$ *is perpendicular to both* $\mathbf{b}$ *and* $\mathbf{c}$ *so it is parallel to* $\mathbf{a}$ *by the mutual perpendicularity. Thus* $\mathbf{x}$ *is the product of two vectors parallel to* $\mathbf{a}$ *and is itself parallel to* $\mathbf{a}$*. Since* $\mathbf{a}$ *and* $\mathbf{c}$ *are perpendicular any vector parallel to both must be the 0 vector. That is*

$$\mathbf{x} = 1.$$

But then the same argument applies for any order of the three vectors and we deduce

$$\mathbf{a} * \mathbf{b} * \mathbf{c} = 1 = \mathbf{b} * \mathbf{a} * \mathbf{c}$$

Cancel the $\mathbf{c}$ *to find*

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b} = \mathbf{b} * \mathbf{a} = \mathbf{b} \times \mathbf{a} = -\mathbf{c}$$

which is a contradiction! We are done.

A6 Let $f : [0,\infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}\, dx$ diverges.

Solution:

Define

$$H(x) = \int_x^{x+1} f(u)du.$$

Note that

$$H'(x) = -(f(x) - f(x+1))$$

and that

$$f(x+1) \le H(x) \le f(x).$$

We are studying the integral

$$I = \int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\, dx = \int_0^\infty \frac{-H'(x)}{H(x)} \frac{H(x)}{f(x)}\, dx.$$

I claim that if the integral converges then

$$\lim_{x\to\infty}\frac{f(x+1)}{f(x)}=1.$$

For if not then there is a $\delta>0$ *and a sequence* $x_n\to\infty$ *such that*

$$\frac{f(x_n+1)}{f(x_n)}<1-\delta$$

By passing to a subsequence we may assume $x_n+2<x_{n+1}$ *for all* n*. Note that*

$$\begin{aligned}\int_{x_n-1}^{x_n+1}\frac{f(y)-f(y+1)}{f(y)}\,dy &= \int_{x_n-1}^{x_n}\frac{f(y)-f(y+1)}{f(y)}\,dy+\int_{x_n}^{x_n+1}\frac{f(y)-f(y+1)}{f(y)}\,dy\\ &= \int_{x_n-1}^{x_n}\frac{f(y)-f(y+1)}{f(y)}\,dy+\int_{x_n-1}^{x_n}\frac{f(y+1)-f(y+2)}{f(y+1)}\,dy\\ &\geq \int_{x_n-1}^{x_n}\frac{f(y)-f(y+1)}{f(y)}\,dy+\int_{x_n-1}^{x_n}\frac{f(y+1)-f(y+2)}{f(y)}\,dy\\ &= \int_{x_n-1}^{x_n}\frac{f(y)-f(y+2)}{f(y)}\,dy\\ &= 1-\int_{x_n-1}^{x_n}\frac{f(y+2)}{f(y)}\,dy\\ &\geq 1-\int_{x_n-1}^{x_n}\frac{f(x_n+1)}{f(x_n)}\,dy\\ &\geq 1-(1-\delta)\\ &= \delta.\end{aligned}$$

Summing over an infinite number of n *shows* $I=\infty$*. So we now assume that*

$$\lim_{x\to\infty}\frac{f(x+1)}{f(x)}=1.$$

Then there is a T *such that for all* $x\geq T$ *we have*

$$f(x+1)/f(x)\geq 1/2.$$

Then for all $x\geq T$ *we have*

$$\frac{f(x+1)}{f(x)}\leq H(x)$$

and so

$$\begin{aligned}I\geq\int_T^\infty\frac{f(y)-f(y+1)}{f(y)}\,dy &= \int_T^\infty\frac{-H'(x)}{H(x)}\frac{H(x)}{f(x)}\,dx\\ &\geq\frac{1}{2}\int_T^\infty\frac{-H'(x)}{H(x)}\frac{H(x)}{f(x)}\,dx\\ &=\frac{1}{2}\left\{-\log(H(x))\right\}|_T^\infty=\infty.\end{aligned}$$

A couple of notes on this. First, I used continuity in asserting the formula for the derivative of H*. I don't seem to have used the fact that* f *is strictly decreasing anywhere I can see.*

B1 Is there an infinite sequence of real numbers $a_1,a_2,a_3,\ldots$ such that

$$a_1^m+a_2^m+a_3^m+\cdots=m$$

for every positive integer m?

Solution:

No. The Cauchy-Schwarz inequality shows

$$m = \left|\sum_1^\infty a_j^m\right| \le \sqrt{\sum_1^\infty a_j^{2l} \sum_1^\infty a_j^{2(m-l)}} = \sqrt{2l(2m-2l)}$$

for all integers $1 \le l \le m-1$. For $m = 4$ and $l = 1$ we get

$$4 \le \sqrt{2(8-2)} = \sqrt{12}$$

which is false. So no such sequence exists.

B2 Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

Solution:

The 3, 4, 5 triangle shows that 3 is possible. I claim 1 and 2 are not possible so the answer is 3. If 1 were possible there would be an example with $A = (0,0)$ and $B = (1,0)$. Put $C = (n, m+1)$. WIthout loss $n > 0$ and $m \ge 0$ (by reflecting any triangle about the x axis if $n < 0$ and then about $x = 1/2$ if $m < 0$). We find that $AC^2 = n^2 + (m+1)^2$ while $BC^2 = n^2 + m^2$. Let $BC = l$. Then $l = BC < AC < AB + BC = l + 1$ which contradicts the assertion that AC is an integer. For $AB = 2$ we put, with no loss, $B = (2,0)$ and $C = (n, 1+m)$ with $n, m \ge 0$. Let $BC = l$. For $m = 0$ we also have $AC = l$ and $l^2 = n^2 + 1$ which requires two perfect squares to differ by 1. This gives $n = 0$ making the points collinear. For $m \ge 1$ we must have $BC < AC < BC + 2$. So if $BC = l$ then $AC = l + 1$ and

$$n^2 + m^2 + 2m + 1 = (l+1)^2$$

while

$$n^2 + m^2 - 2m + 1 = l^2.$$

Subtracting gives $2l + 1 = 4m$. But $2l + 1$ is odd while $4m$ is even. So the smallest possible value is 3.

B3 There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

Solution:

If we start with fewer than i balls in box i for each i then no moves are possible. If we have fewer than $\sum_1^{2010}(i-1) = 2009 \cdot 2010/2 \equiv m$ balls then we can put 0 balls in box 1, 1 in box 2 and so on an use up all the balls while ensuring that there are fewer than i balls in box i for every i. This means that n does not work unless

$$2010n \ge 2009 \cdot 2010/2$$

or $n \ge 1005$. Now suppose that $n \ge 1005$. I claim that it is possible to put all the balls into box 1. If so then they can clearly be redistributed 1 at a time to put exactly n in each box. Since the number of balls exceeds m there is a box $i > 1$ with at least i balls in it. For each such box move batches of i balls to box 1 until there are fewer than i balls in each box for $i > 1$. There are now at least 2 balls in box 1. Put balls one at a time from box 1 into box 2 until there are an even number of balls in box 2. Then move them all, 2 at a time, to box 1; this leaves box 2 empty. Either box 3 is empty or there are enough balls in box 1 to fill box 3 to a multiple of 3. Move them all, 3 at a time, to box 1. Now boxes 2 and 3 are empty. Counting up the balls in boxes with $i \ge 4$ shows that there are at least 3 balls in box 1 so we can move them to box 4 one at a time to get a multiple of 4 balls in box 4. Empty box 4 into box 1. Now suppose you have emptied boxes 2 through $j-1$. The number of balls in boxes j through 2010 is at most

$$\sum_j^{2010}(i-1) = m - j(j-1)/2$$

so there are at least $j(j-1)/2 > j$ balls in box 1. We can then move balls to box j from box 1 to make a multiple of j balls and then empty box j. This shows inductively that we may put all the balls into box 1. Conclusion: we can reach the desired distribution for all $n \ge 1005$.

B4 Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

Solution:

The solution is that p and q must both be linear and, if $p(x) = a + bx$ and $q(x) = c + dx$ then

$$ad - bc = 1.$$

So we will prove this.

If p, q is any pair solving the problem then so are $p, q + ap$ and $p + aq, q$ for any constant a. Thus we may assume that if there is any solution in which either polynomial has degree larger than 1 there is a solution in which both have degree $d > 1$. Then write

$$p(x) = \sum_0^d a_j x^j$$

and

$$q(x) = \sum_0^d b_j x^j$$

with neither of a_d or b_d equal to 0. Then

$$\begin{aligned} p(x)q(x+1) - q(x)p(x+1) &= \sum_0^d \sum_0^d (a_j b_k - a_k b_j) x^j (x+1)^k \\ &= \sum_0^d \sum_0^d (a_j b_k - a_k b_j) \sum_{l=0}^k \binom{k}{l} x^{j+l} \\ &= \sum_{r=0}^{2d} x^r \sum_0^d \sum_0^d \sum_{l=0}^k (a_j b_k - a_k b_j) \binom{k}{l} 1(j + l = r) \\ &\equiv \sum_{r=0}^{2d} c_r x^r \end{aligned}$$

where

$$c_r = \sum_0^d \sum_0^d \sum_{l=0}^k (a_j b_k - a_k b_j) \binom{k}{r-j}$$

and the binomial coefficient is 0 if either $r - j < 0$ or $r - j > k$. We note terms with $j = k$ automatically vanish and write

$$c_r = \sum_{0 \le j < k \le d} (a_j b_k - a_k b_j) \left\{ \binom{k}{r-j} - \binom{j}{r-k} \right\}.$$

Any solution must have $c_{2d} = \cdots = c_1 = 0$. Putting $r = 2d$ we see both binomial coefficients vanish unless

$$2d - k \le j \text{ and } 2d - j \le k.$$

These just reduce to $j + k \ge 2d$ which, for $j < k \le d$ is impossible so $c_{2d} = 0$. For $r = 2d - 1$ we find $j = d - 1$ and $k = d$ and the two binomial coefficients reduce to 1 so $c_{2d-1} = 0$ is automatic. For $r = 2d - 2$ we have the term $j = d - 2, k = d$ for which

$$\binom{k}{r-j} = \binom{d}{d} = 1 \text{ while } \binom{j}{r-k} = \binom{d-2}{d-2} = 1$$

giving a 0 and the term $j = d-1, k = d$ for which

$$\binom{k}{r-j} = \binom{d}{d-1} = d \text{ while } \binom{j}{r-k} = \binom{d-1}{d-2} = d-1$$

giving the requirement

$$(a_{d-1}b_d - a_d b_{d-1})(d - (d-1)) = 0$$

This simplifies to

$$a_{d-1}b_d - a_d b_{d-1} = 0.$$

Notice that for $d = 1$ the binomial coefficient $\binom{d-1}{d-2}$ is actually 0 and we do not deduce this last equation.
We now argue that $a_j b_k - a_k b_j = 0$ for all $0 \le j < k \le d$. If so then we have shown

$$p(x)q(x+1) - q(x)p(x+1) \equiv 0$$

and this is not a solution of our problem. In fact since $b_d \neq 0$ and $a_d \neq 0$ it is enough to do the case $k = d$. Now we do induction on j starting at $j = d-1$ which we have just done. Suppose we have established $a_j b_d - a_d b_j = 0$ for all $j_0 < j \le d-1$. Then for any pair $j_0 < j < k \le d-1$ we have

$$a_j = b_j a_d / b_d \text{ and } b_k = a_k b_d / a_d$$

which multiply together to show

$$a_j b_k - a_k b_j = 0$$

for all $j_0 < j < k \le d-1$. The formula for c_r now simplifies to

$$c_r = \sum_{0 \le j \le j_0; j < k \le d} (a_j b_k - a_k b_j) \left\{ \binom{k}{r-j} - \binom{j}{r-k} \right\}.$$

Take $r = d + j_0$. The coefficient $\binom{k}{d+j_0-j}$ will be 0 unless

$$0 \le d + j_0 - j \le k \le d$$

giving $j + k \ge d + j_0$. The coefficient $\binom{j}{d+j_0-k}$ will be 0 except in the same circumstances. But the restriction in the sum defining c_r now shows $j + k \le d + j_0$ so we must have $j = j_0$ and $k = d$. The corresponding binomial coefficient difference becomes

$$\binom{k}{r-j} - \binom{j}{r-k} = \binom{d}{d} - \binom{j_0}{j_0} = 0.$$

So $c_{d+j_o} = 0$ is automatic.
Then take $r = d - 1 + j_0$. The coefficient $\binom{k}{d-1+j_0-j}$ will be 0 unless

$$0 \le d - 1 + j_0 - j \le k \le d$$

giving $j + k \ge d + j_0 - 1$. The coefficient $\binom{j}{d-1+j_0-k}$ will be 0 except in the same circumstances. But the restriction in the sum defining c_r now shows $j + k \le d + j_0$ so there are the following terms to consider: $j = j_0, k = d$, $j = j_0 - 1, k = d$ and $j = j_0, k = d-1$. In the latter two cases the binomial coefficients both simplify to 1. We are thus left with

$$0 = c_{d+j_0-1} = (a_{j_0} b_d - a_d b_{j_0}) \left\{ \binom{d}{d + j_0 - 1 - j_0} - \binom{j_0}{d + j + 0 - 1 - d} \right\}$$

which simplifies to

$$0 = c_{d+j_0-1} = (a_{j_0} b_d - a_d b_{j_0})(d - j_0)$$

It follows that $a_{j_0} b_d = a_d b_{j_0}$ completing the induction.

B5 Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?

Solution:

No. Since f is strictly increasing f' must be non-negative and strictly increasing. Thus $f'(0) > 0$. If $f(0) \le 0$ then, since f is strictly increasing, we have $f'(0) = f(f(0)) \le f(0) \le 0$ a contradiction. Thus $f(0) > 0$. Next for $x \ge 0$ we have

$$\begin{aligned} f(x+1) &= f(x) + \int_x^{x+1} f'(u)du \\ &= f(x) + \int_x^{x+1} f(f(u))du \\ &\ge \int_x^{x+1} f(f(x))du \\ &= f(f(x)) \end{aligned}$$

Since f is strictly increasing we deduce that for all $x \ge 0$

$$f(x) \le x + 1.$$

On the other hand $\lim_{x\to\infty} f(x) = \infty$ because

$$f(x) = f(0) + \int_0^x f'(u)du \ge x f'(0).$$

But then $\lim_{x\to\infty} f'(x) = \infty$ and so $f'(x) > 2$ for all large enough x say $x \ge x_0$. For $x > x_0$ we must have

$$f(x) \ge f(x_0) + 2(x - x_0)$$

implying the contradiction

$$f(x_0) + 2(x - x_0) \le x + 1$$

for all large x.

B6 Let A be an $n \times n$ matrix of real numbers for some $n \ge 1$. For each positive integer k, let $A^{[k]}$ be the matrix obtained by raising each entry to the k^{th} power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n+1$, then $A^k = A^{[k]}$ for all $k \ge 1$.

Solution:

Let

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + \sum_0^{n-1} c_j \lambda_j$$

be the characteristic polynomial of A. The Cayley-Hamilton theorem shows

$$0 = A^n + \sum_0^{n-1} c_j A^j$$

Multiply by A to see

$$0 = A^{n+1} + \sum_0^{n-1} c_j A^{j+1} = A^{[n+1]} + \sum_0^{n-1} c_j A^{[j+1]}.$$

and for each pair i, l we therefore have

$$0 = A_{il}^{n+1} + \sum_0^{n-1} c_j A_{il}^{j+1}.$$

We now prove

$$A^{n+r} = A^{[n+r]}$$

by induction on r. It is given for $r = 1$. Assume it is established for all $r < r_0$. Then multiplying the identity above by A^{r_0-1} gives

$$0 = A^{n+r_0} + \sum_0^{n-1} c_j A^{j+r_0} = A^{n+r_0} + \sum_0^{n-1} c_j A^{[j+r_0]}.$$

On the other hand if we multiply the i, l identity by $A_{il}^{r_0-1}$ gives

$$0 = A_{il}^{n+r_0} + \sum_0^{n-1} c_j A_{il}^{j+r_0}$$

which means

$$0 = A^{[n+r_0]} + \sum_0^{n-1} c_j A^{[j+r_0]}.$$

Comparison the two identities shows

$$A^{n+r_0} = A^{[n+r_0]}$$

finishing the induction. Notice that we needed $n + 1$ not n because of the intercept terms.