The 66th William Lowell Putnam Mathematical Competition
Saturday, December 3, 2005

NOTE: These are Richard Lockhart's "solutions" meaning no-one knows if they are right.

A1 Show that every positive integer is a sum of one or more numbers of the form 2^r3^s, where r and s are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)

Proof: *Let $P(n)$ denote the assertion that n admits such a decomposition. It is easy to see that $P(1)$ and $P(2)$ are true. If the proposition is not true consider the least value of n for which the assertion is not true. This value of n must be odd since if n were even and each member of a decomposition of $n/2$ were doubled we would get a decomposition of n. Now find the largest r such that $3^r \le n$ and let $m = n - 3^r$. If $m = 0$ we have $n = 3^r$ which is the desired decomposition. Suppose that $m > 0$. Then since n is odd and 3^r is odd we see that m is even, say $m = 2k$. Since $P(k)$ is true we may find a decomposition of k. Double each member of that decomposition to get a decomposition of m for which each term is even. Since these terms are even none of them divides 3^r. Suppose to derive a contradiction that 3^r divided one of the terms in the decomposition of m. That term would be $l3^r$ and since the term in the decomposition of m is even we would have $l \ge 2$. But then $n \ge l3^r + 3^r \ge 3^{r+1}$ which contradicts the construction of r finishing the proof.*

A2 Let $\mathbf{S} = \{(a,b)|a = 1,2,\ldots,n, b = 1,2,3\}$. A *rook tour* of $\mathbf{S}$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that

(i) $p_i \in \mathbf{S}$,

(ii) p_i and p_{i+1} are a unit distance apart, for $1 \le i < 3n$,

(iii) for each $p \in \mathbf{S}$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1,1)$ and end at $(n,1)$?

(An example of such a rook tour for $n = 5$ was depicted in the original.)

Solution: *Fix a tour. Let r_1 denote the first column in which the tour goes up to row 2. It is easily seen that there cannot be a downward step until all squares (j,k) with $j \le r_1$ have been visited and that the last of these squares visited must be $(r_1, 3)$. Now let $r_2 > r_1$ be the first column in which the tour goes down to row 2 from row 3. Again the tour can follow only one path until all squares (j,k) with $j \le r_2$ have been visited terminating with $(r_2, 1)$. Continuing in this way we see that the number of tours is equal to the number of sequences $1 \le r_1 < r_2 < \cdots < r_{2k} = n$. This is the number of subsets of odd cardinality of $\{1,\ldots,n-1\}$ or*

$$\binom{n-1}{1} + \binom{n-1}{3} + \cdots + \binom{n-1}{2m+1}$$

where $2m+1$ is the largest odd integer less than n. If $n-1$ is odd then there is a one to one match between subsets of odd cardinality and subsets of even cardinality so the number of subsets of odd cardinality is half the number of subsets. That is, there are $2^{n-1}/2 = 2^{n-2}$. For $n-1$ even a subset of odd cardinality can be obtained either by taking a subset of even cardinality of $\{1,\ldots,n-2\}$ and adding $n-1$ or by taking a subset of odd cardinality of $\{1,\ldots,n-2\}$ and not adding $n-1$. This produces $2^{n-3} + 2^{n-3} = 2^{n-2}$ subsets of odd cardinality of $\{1,\ldots,n-1\}$ We thus see that there are 2^{n-2} such tours. Notice that for $n = 1$ there are no such tours.

A3 Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

Proof: *If $h = \log g$ then $h' = g'/g$ has zeroes exactly where g' has zeroes. Since p is a polynomial there is no loss in writing*

$$p(z) = \prod_1^n (z - a_i)$$

where each a_i has modulus 1. Thus

$$h = \sum \log(z - a_i) - (n/2)\log(z)$$

and

$$h' = \sum \{1/(z-a_i) - 1/(2z)\}$$
$$= \sum \frac{z+a_i}{2z(z-a_i)}$$

Notice that $h'(z) = 0$ *if and only if*

$$0 = \sum \frac{z+a_i}{z-a_i}$$
$$= \sum \frac{(z+a_i)(\bar{z}-\bar{a}_i)}{(z-a_i)(\bar{z}-\bar{a}_i)}$$
$$= \sum \frac{|z|^2-|a_i|^2}{|z-a_i|^2} + \sum \frac{a_i\bar{z}-z\bar{a}_i}{|z-a_i|^2}$$

Notice that the first term is real and the second is imaginary. Thus any root z *of* $h'(z) = 0$ *must make both terms 0. But*

$$\sum \frac{|z|^2-|a_i|^2}{|z-a_i|^2} = \sum \frac{|z|^2-1}{|z-a_i|^2}$$

is strictly positive if $|z|^2 > 1$ *and strictly negative is* $|z|^2 < 1$ *so that unless* $|z| = 1$*, we cannot have* $h'(z) = 0$.

A4 Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

Solution: *By permuting the rows and columns of* H *we may assume that the top* $a \times b$ *corner of* H *is all 1s. Let* $\mathbf{z}_i$ *denote the* i*th row of* H *and let* $\mathbf{x}_i$ *denote the first* b *entries in* $\mathbf{z}_i$ *and* $\mathbf{y}_i$ *denote the last* $n-i$ *entries in* $\mathbf{z}_i$*. Let* $Z = \sum_{i=1}^{a} \mathbf{z}_i$ *with* X *and* Y *defined analogously. Notice that*

$$||Z||^2 = ||X||^2 + ||Y||^2 \geq ||X||^2$$

Since $\mathbf{z}_1, \ldots, \mathbf{z}_a$ *are orthogonal and each has squared length equal to* n *we find*

$$||Z||^2 = na$$

On the other hand X *is a vector all of whose entries are equal to* a *so that*

$$||X||^2 = ba^2$$

This gives

$$na \geq ba^2$$

or, cancelling a*,*

$$ab \leq n$$

A5 Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\, dx.$$

Solution: *Let* I *be the integral in question. Substitute* $x = (1-y)/(1+y) = 2/(1+y) - 1$ *and* $dx = -2/(1+y)^2 dy$ *to get*

$$I = 2\int_0^1 \ln(2/(1+y))/(1+(1-y)^2/(1+y)^2)\frac{dy}{(1+y)^2}$$
$$= \ln(2)\int_0^1 \frac{dy}{1+y^2} - I$$
$$= \ln(2)\left\{\tan^{-1}(1) - \tan^{-1}(0)\right\} - I$$
$$= \pi\ln(2)/4 - I$$

Solve to get

$$I = \pi \ln(2)/8$$

A6 Let n be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are n randomly, independently and uniformly, chosen points on a circle. Consider the convex n-gon whose vertices are P_i. What is the probability that at least one of the vertex angles of this polygon is acute?

Solution: *If points P, Q and R are on the circle in that order proceeding clockwise then the angle PQR will be acute if and only if R is more than half way round the circle from P. Thus there cannot be more than 2 strictly acute angles and if there are two strictly acute angles they must be adjacent. For $i \neq j$ let $A(i,j)$ denote the following event:*

- *Points P_i, P_j occur in sequence proceeding counterclockwise around the circle.*
- *The angles at P_i and P_j are acute.*

Similarly for $i \neq j$ let $B(i,j)$ denote the following event:

- *Points P_i, P_j occur in sequence proceeding counterclockwise around the circle.*
- *The angle at P_i is acute and there are no other acute angles.*

The event described in the question is the union over all choices of $i \neq j$ of all $A(i,j)$ and over all $i \neq j$ all different of the events $B(i,j)$. These events are all disjoint. By symmetry the probabilities in question do not depend on i, j, k. The probability desired is thus

$$n(n-1)P(A(1,2)) + n(n-1)P(B(1,2))$$

Consider first $P(A(1,2))$. By conditioning on the location of P_1 we have assume that P_1 occurs at the angle $\theta = 0$ and identify points on the circle by their angles θ_i measured in the range $-\pi < \theta \leq \pi$ running counterclockwise around the circle. If P_2 occurs at the point θ_2 in the range $-\pi < \theta_2 < 0$ then $A(1,2)$ occurs provided all other points occur in the interval $(\theta_2, 0)$. The conditional probability of this, given θ_2, is

$$\left(\frac{-\theta_2}{2\pi}\right)^{n-2}$$

If, on the other hand P_2 occurs at θ_2 in $(0, \pi]$ then $A(1,2)$ occurs provided all the other points are in the interval $(-\pi, \theta_2 - \pi)$. This has conditional probability

$$\left(\frac{\theta_2}{2\pi}\right)^{(n-2)}$$

It follows that

$$\begin{aligned} P(A(1,2)) &= \frac{1}{(2\pi)^{n-1}} \int_{-\pi}^{0} (-\theta_2)^{n-2} d\theta_2 \\ &\quad + \frac{1}{(2\pi)^{n-1}} \int_{0}^{\pi} (\theta_2)^{n-2} d\theta_2 \\ &= 2^{-(n-1)}/(n-1) + 2^{-(n-1)}/(n-1) \\ &= 2^{-(n-2)}/(n-1) \end{aligned}$$

On the other hand to compute $P(B(1,2))$ we may assume without loss that $\theta_1 = 0$. In order for $B(1,2)$ to occur we must have

1. *point θ_2 must be in the interval $(0, \pi]$,*
2. *all remaining points must be in the half circle starting at θ_2 and proceeding clockwise.*
3. *there must be at least one point in the interval $(\theta_2, \pi]$.*
4. *there must be at least two points in the interval $(-\pi, \theta_2 - \pi)$*

Given a θ_2 satisfying condition 1 the probability that conditions 2, 3 and 4 are satisfied is the probability that 2 is satisfied minus the probability that all $n-2$ remaining points lie in $(-\pi, \theta_2 - \pi)$ minus the probability that all $n-2$ points lie in $(\theta_2, \pi]$ minus the probability that exactly 1 point lies in $(-\pi, \theta_2 - \pi)$ and $n-3$ lie in $(\theta_2, \pi]$. This conditional probability is

$$2^{-(n-2)} - \left(\frac{\theta_2}{2\pi}\right)^{n-2} - \left(\frac{\pi - \theta_2}{2\pi}\right)^{n-2} - (n-2)\left(\frac{\theta_2}{2\pi}\right)\left(\frac{\pi - \theta_2}{2\pi}\right)^{n-3}$$

Multiply by the density of θ_2 and integrate from 0 to π. Make the substitution $\theta_2/(2\pi) = u$ in the integrals to get

$$\begin{aligned} P(B(1,2)) &= 2^{-(n-1)} - \int_0^{1/2} u^{n-2} du - \int_0^{1/2} (1/2-u)^{n-2} du - (n-2) \int_0^{1/2} u(1/2-u)^{n-3} du \\ &= 2^{-(n-1)} - 2^{-(n-1)}/(n-1) - 2^{-(n-1)}/(n-1) - (n-2)\int_0^{1/2} (1/2-u)u^{n-3} du \\ &= 2^{-(n-1)}(1 - 2/(n-1)) - (n-2)\left(2^{-(n-1)}/(n-2) - 2^{-(n-1)}/(n-1)\right) \end{aligned}$$

The desired probability is then

$$n(n-1)\left\{2^{-(n-2)}/(n-1) + 2^{-(n-1)}\left(1 - 2/(n-1) - 1 + (n-2)/(n-1)\right)\right\}$$

which becomes

$$2^{-(n-1)}\left\{2n - 2n + n(n-2)\right\} = n(n-2)2^{-(n-1)}.$$

B1 Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν.)

Solution: *If $a = n + \epsilon$ with $0 \le \epsilon < 1$ then $\lfloor a \rfloor = n$ and $\lfloor 2a \rfloor$ is either $2n$ or $2n+1$. That is, either*

$$\lfloor 2a \rfloor = 2\lfloor a \rfloor$$

or

$$\lfloor 2a \rfloor = 2\lfloor a \rfloor + 1$$

Put $P(x, y) = (y - 2x)(y - (2x+1))$. Evaluated at $x = \lfloor a \rfloor$ and $y = \lfloor 2a \rfloor$ either the first term in P or the second must be 0.

B2 Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

Solution: *Notice that*

$$\sum_{ij} k_i k_j^{-1} = n + \sum_{i \neq j} ki/k_j$$

Since for $x > 0$ we have $x + 1/x \ge 2$ with equality only at $x = 1$ we see that $k_i/k_j + k_j/k_i \ge 2$ and so

$$\sum_{ij} k_i k_j^{-1} \ge n^2$$

with the inequality strict unless all the k_i are equal.

If all the k_i are equal to say m then we find $nm = 5n - 4$ and find that n must divide 4 so that n is one of 1,2 or 4. For $n = 1$ this gives $m = 1$ and $k_1 = 1$ which is a solution. For $n = 2$ we get $m = 3$ which does not give a solution. For $n = 4$ we get $m = 4$ and this is a solution: $k_1 = k_2 = k_3 = k_4 = 4$.

I claim there are no other solutions. If $a = \sum k_i = 5n - 4$ *and* $b = \sum k_i^{-1} = 1$ *then*

$$ab = 5n - 4 > n^2$$

or

$$n^2 - 5n + 4 < 0$$

The left hand side factors as $(n-1)(n-4)$ *which is negative only for* $n = 2$ *or* $n = 3$. *For* $n = 2$ *we would need 2 integers summing to 6. The possibilities are* $(1,5)$, $(2,4)$ *and* $(3,3)$ *none of which are solutions. For* $n = 3$ *we would need 3 integers summing to 11. Clearly the smallest cannot be 1 since the reciprocals would then add to more than 1. The possible triples are then, after sorting,* $(2,2,7)$, $(2,3,6)$, $(2,4,5)$, $(3,3,5)$ *and* $(3,4,4)$. *Of these only* $(2,3,6)$ *solves both equations. Thus the complete list of solutions is*

$$n = 1, k_1 = 1$$

$$n = 4, k_1 = k_2 = k_3 = k_4 = 1$$

and

$$n = 3, \{k_1, k_2, k_3\} = \{2, 3, 6\}$$

(In the last of these there are 6 possible orders for the numbers 2, 3 and 6 to be assigned.)

B3 Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

Solution: *If* $f(x) = cx^r$ *for some* $c > 0$ *then*

$$f(x)f'\left(\frac{a}{x}\right) = c^2 r(a/x)^(r-1)x^r = c^2 r a^{r-1} x = x$$

provided $r > 0$ *and we choose* a *by*

$$a = \left(\frac{1}{c^2 r}\right)^{1/(r-1)}$$

On the other hand suppose f *and* a *satisfy the identity. Notice that the right hand side of the original identity is differentiable and so* f' *is differentiable. Replace* x *by* a/x *to get*

$$f\left(\frac{a}{x}\right) = \frac{a}{xf'(x)}$$

Now define $g(x) = xf'(x)$ *and differentiate to get*

$$-\frac{a}{x^2}f'\left(\frac{a}{x}\right) = -\frac{ag'(x)}{g^2(x)}$$

But then

$$-\frac{a}{x^2}\frac{x}{f(x)} = -\frac{ag'(x)}{g^2(x)}$$

Rearrange to get

$$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)}$$

Integrate to get

$$\ln(f(x)) = \ln(g(x)) + c_1$$

or

$$f(x) = c_2 x f'(x)$$

Thus

$$c_2 \ln(f(x)) = \ln(x) + c_3$$

or

$$f(x) = c_4 x_5^c$$

Thus such a constant a exists if and only if f has the form

$$f(x) = cx^r$$

for some $r > 0$.

B4 For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m,n) = f(n,m)$.

Proof: *Fix m and n. Suppose that $x_1, \ldots, x_m$ is a sequence of integers with $|x_1|+|x_2|+\cdots+|x_m| \leq n$. We will produce $y_1, \ldots, y_n$ a sequence of integers with $|y_1| + |y_2| + \cdots + |y_n| \leq m$. We will show that distinct x's map to distinct y. This will show $f(n,m) \leq f(m,n)$. Reversing the roles of m and n in our construction we would establish the opposite inequality which would complete the proof.*

Let $S_j = |x_1|+\cdots+|x_j|$ for $j = 1, \ldots, m$. Notice that each S_j is an integer in the range 1 to n. Let $1 \leq j_1 < \cdots < j_r \leq n$ be the distinct non-zezro values of $S_1, \ldots, S_m$. Let $k_1 < \cdots < k_r$ be the indexes of the non-zero entries in the sequence $x_1, \ldots, x_m$. The non-zero entries in the sequence y will be $y_{j_1}, \ldots, y_{j_r}$. We take $|y_{j_1}| = k_1$, $|y_{j_2}| = k_2 - k_1$ and so on to $|y_{j_r}| = k_r - k_{r-1}$. Put all other $y_i = 0$. Notice that $|y_1| + \cdots + |y_n| = k_r \leq m$. If $x_{k_i} > 0$ take $y_{j_i} > 0$; if $x_{k_i} < 0$ take $y_{j_i} < 0$. Evidently y satisfies the requirements to be counted in $f(m,n)$.

Now imagine repeating the process described beginning with y. If $S_j^* = |y_1| + \cdots |y_j|$ then the distinct values of S_j^* are precisely $k_1 < \cdots < k_r$ and the indices of the non-zero entries in y are $j_1, \ldots, j_r$. It is not hard to see, then, that the construction just given produces the original x showing that the map from x to y is injective.

B5 Let $P(x_1, \ldots, x_n)$ denote a polynomial with real coefficients in the variables $x_1, \ldots, x_n$, and suppose that

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) P(x_1, \ldots, x_n) = 0 \quad \text{(identically)}$$

and that

$$x_1^2 + \cdots + x_n^2 \text{ divides } P(x_1, \ldots, x_n).$$

Show that $P = 0$ identically.

Proof: *Let Δ denote the Laplacian operator:*

$$\Delta A = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) A$$

for any function $A = A(x_1, \ldots, x_n)$. Any polynomial can be written in the form

$$\sum_{k=0}^{N} T_k$$

where T_k has the form

$$T_k = \sum_{i_1 \cdots i_k} a(i_1 m \ldots, i_k) x_{i_1} \cdots x_{i_k}$$

Notice that the Laplacian operating on any T_k produces an object of the form

$$\sum_{j_1 \cdots j_{k-2}} b(j_1, \ldots, j_{k-1}) x_{j_1} \cdots x_{j_{k-2}}.$$

Thus if there is a counterexample there is one which can be written in the form

$$P = R^m S$$

where $R = x_1^2 + \cdots + x_n^2$ *and*

$$S = \sum_{i_1 \cdots i_k} a(i_1 m \ldots, i_k) x_{i_1} \cdots x_{i_k}$$

and S *is not divisible by* R.

We will need the following claims:

1. *For any* A *and* B *we have*

$$\Delta(AB) = B\Delta A + 2(\nabla A)^{\nabla} B + A\Delta B$$

   where ∇A *denotes the gradient of* A *written as a column vector and the superscript is a matrix transpose.*

2. *For any* $m > 1$ *we have*

$$\Delta R^m = \{2nm + 4m(m-1)\} R^{m-1}$$

3. *We have*

$$\mathbf{x}^t \nabla S = kS$$

   where $\mathbf{x}^t = (x_1, \ldots, x_n)$.

4. *We have*

$$\nabla R^m = 2mR^{m-1}\mathbf{x}.$$

For the putative counterexample we then have

$$\begin{aligned}
0 &= \Delta R^m S \\
&= S\Delta R^m + 2(\nabla R^m)^t \nabla S + R^m \Delta S \\
&= \{2nm + 4m(m-1)\} R^{m-1} S + 4mR^{m-1}\mathbf{x}^t \nabla S + R^m \Delta S \\
&= \{2nm + 4m(m-1)\} R^{m-1} S + 4mkR^{m-1} S + R^m \Delta S \\
&= R^{m-1} \left[\{2nm + 4m(m-1) + 4mk\} S + R\Delta S\right]
\end{aligned}$$

Since this is an identity we may cancel the factor R^{m-1} *which is positive everywhere except at* $\mathbf{x} = 0$ *to get*

$$0 = \{2nm + 4m(m-1) + 4mk\} S + R\Delta S$$

If ΔS *is not identically 0 then we have a factorization of* S *with* R *as a factor. This contradicts the definition of* m. *If* $\Delta S = 0$ *we find* $S = 0$. *In either case we conclude there is no such counterexample.*

It remains to establish the claims. The first assertion is an easy consequence of the product rule applied twice. For the second write

$$\begin{aligned}
\Delta R^m &= \sum_{l=1}^{n} \frac{\partial^2}{\partial x_l^2} R^m \\
&= \sum_{l=1}^{n} \frac{\partial}{\partial x_l} 2mR^{m-1} x_l \\
&= \sum_{l=1}^{n} 2m(2(m-1)R^{m-2} x_l^2 + R^{m-1}) \\
&= 4m(m-1)R^{m-2} \sum_{l=1}^{n} x_l^2 + 2mnR^{m-1} \\
&= \{2nm + 4m(m-1)\} R^{m-1}
\end{aligned}$$

as asserted.

For the third claim write

$$\begin{aligned}
\mathbf{x}^t \nabla S &= \sum_{l=1}^{n} x_l \frac{\partial}{\partial x_l} \sum_{i_1 \cdots i_k} a(i_1, \ldots, i_k) x_{i_1} \cdots x_{i_k} \\
&= \sum_{l=1}^{n} x_l \sum_{i_1 \cdots i_k} a(i_1, \ldots, i_k) \sum_{r=1}^{k} \prod_{s \neq r} x_{i_s} \frac{\partial x_{i_r}}{\partial x_l} \\
&= \sum_{l=1}^{n} x_l \sum_{i_1 \cdots i_k} a(i_1, \ldots, i_k) \sum_{r=1}^{k} \prod_{s \neq r} x_{i_s} 1(i_r = l) \\
&= \sum_{i_1 \cdots i_k} a(i_1, \ldots, i_k) \sum_{r=1}^{k} \prod_{s \neq r} x_{i_s} \sum_{l=1}^{n} x_l 1(i_r = l) \\
&= \sum_{i_1 \cdots i_k} a(i_1, \ldots, i_k) \sum_{r=1}^{k} \prod_{s \neq r} x_{i_s} x_{i_r} \\
&= \sum_{i_1 \cdots i_k} a(i_1, \ldots, i_k) \sum_{r=1}^{k} \prod_{s} x_{i_s} \\
&= k \sum_{i_1 \cdots i_k} a(i_1, \ldots, i_k) \prod_{s} x_{i_s} \\
&= kS
\end{aligned}$$

The fourth claim is an elementary calculation concluding the proof.

B6 Let S_n denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi) + 1} = (-1)^{n+1} \frac{n}{n+1}.$$

Proof: *Define*

$$a_{n,k} = \sum_{\pi : \nu(\pi) = k} \sigma(\pi)$$

If permutation π fixes k elements of $1, \ldots, n$ then restricted to the remaining $n - k$ elements it fixes none. We therefore have

$$a_{n,k} = \binom{n}{k} a_{n-k,0}$$

It is well known that exactly half the permutations of $\{1, \ldots, n\}$ are odd for $n > 1$. That is,

$$a_{n,0} + \cdots + a_{n,n} = \sum_{\pi} \sigma(\pi) = 0$$

for $n \geq 2$. These two facts permit us to prove by induction first the claim

$$a_{n,0} = (-1)^{n-1}(n-1)$$

for all $n \geq 2$, and then the desired result.

Suppose the claim has been established for all positive integers strictly less than n. (Notice that there are no permutations with $\nu(\pi) = n-1$; we take $a_{n,n-1} = 0$ for convenience.) Then

$$\begin{aligned}
a_{n,0} &= -\sum_{j=1}^{n-1} a_{nj} - a_{n,n} \\
&= -\sum_{j=1}^{n-1} \binom{n}{j} (-1)^{n-j-1}(n-j-1) - 1 \\
&= -\sum_{j=1}^{n-1} \frac{n!}{j!(n-j)!}(-1)^{n-j-1}(n-j) - \sum_{j=1}^{n-1} \frac{n!}{j!(n-j)!}(-1)^{n-j} - 1 \\
&= -\sum_{j=1}^{n-1} \frac{n \cdot (n-1)!}{j!(n-1-j)!}(-1)^{n-1-j} - \sum_{j=1}^{n-1} \frac{n!}{j!(n-j)!}(-1)^{n-j} - 1 \\
&= -n \left\{ \sum_{0}^{n-1} \binom{n-1}{j} (-1)^{n-1-j} - (-1)^{n-1} \right\} - \left\{ \sum_{0}^{n} \binom{n}{j} (-1)^{n-j} - (-1)^n - 1 \right\} - 1 \\
&= -n \left\{ 0 - (-1)^{n-1} \right\} + \left\{ 0 - (-1)^{n-1} + 1 \right\} - 1 \\
&= (-1)^{n-1}(n-1)
\end{aligned}$$

which establishes the claim by induction.

To finish the problem write

$$\begin{aligned}
\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi)+1} &= \sum_{k=0}^{n} \frac{1}{k+1} \sum_{\nu(\pi)=k} \sigma(\pi) \\
&= \sum_{k=0}^{n} \frac{1}{k+1} a_{n,k} \\
&= \sum_{k=0}^{n-2} \frac{n!}{(k+1)k!(n-k)!}(-1)^{n-k-1}(n-k-1) + \frac{1}{n+1} \\
&= \frac{1}{n+1} + \sum_{k=0}^{n-2} \frac{n!}{(k+1)!(n-k-1)!}(-1)^{n-k-1} + \frac{1}{n+1} \sum_{k=0}^{n-2} \frac{(n+1)!}{(k+1)!(n-k)!}(-1)^{n+1-k-1} \\
&= \frac{1}{n+1} + \sum_{j=1}^{n-1} \binom{n}{j} (-1)^{n-j} + \frac{1}{n+1} \sum_{j=1}^{n-1} \binom{n+1}{j} (-1)^{n+1-j} \\
&= \frac{1}{n+1} + \left\{ \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} - (-1)^n - 1 \right\} \\
&\quad + \frac{1}{n+1} \left\{ \sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^{n+1-j} - (-1)^{n+1} - (-1)(n+1) - 1 \right\} \\
&= \frac{1}{n+1} + \{0 - (-1)^n - 1\} + \frac{1}{n+1} \left\{ 0 - (-1)^{n+1} + n \right\} \\
&= \frac{1}{n+1} + (-1)^{n+1} - 1 - \frac{1}{n+1} \left\{ (-1)^{n+1} - n \right\} \\
&= (-1)^{n+1} \frac{n}{n+1}
\end{aligned}$$