A1 Basketball star Shanille O’Keal’s team statistician keeps track of the number, \( S(N) \), of successful free throws she has made in her first \( N \) attempts of the season. Early in the season, \( S(N) \) was less than 80% of \( N \), but by the end of the season, \( S(N) \) was more than 80% of \( N \). Was there necessarily a moment in between when \( S(N) \) was exactly 80% of \( N \)?

\[ A: \text{Yes. Let } S(n) \text{ be the number of successes in the first } n \text{ throws. Then } F(n) = n - S(n) \text{ is the number of failures. Put } \]

\[ W(n) = S(n) - 4F(n) \]

and note that the success percentage is less than 80, exactly 80 or more than 80 according as \( W(n) \) is negative, 0 or positive. Notice too that either \( W(n + 1) = W(n) + 1 \) or \( W(n + 1) = W(n) - 4 \). Let \( M \leq N \) be the least \( n > 1 \) such that \( W(n) > 0 \). Such an \( M \) exists from the assumptions. Now \( W(M - 1) \leq 0 \) by definition of \( M \) and so

\[ W(M) \leq W(M - 1) + 1 \leq 1 \]

Since \( W(M) \) is an integer and \( W(M) > 0 \) we find \( W(M) = 1 \). This shows

\[ W(M - 1) \geq W(M) - 1 \geq 0 \]

and so \( W(M - 1) = 0 \). That is, at toss \( M - 1 \) the success rate was exactly 4/5.

A2 For \( i = 1, 2 \) let \( T_i \) be a triangle with side lengths \( a_i, b_i, c_i \), and area \( A_i \). Suppose that \( a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2 \), and that \( T_2 \) is an acute triangle. Does it follow that \( A_1 \leq A_2 \)?

\[ A: \text{Yes. Let } \alpha_i, \beta_i, \gamma_i \text{ be the angles opposite sides } a_i, b_i, c_i \text{ respectively. Since the two sets of angles have the same sum there is an angle on triangle 2 which is larger than the corresponding angle on triangle 1. Without loss suppose } \alpha_2 \geq \alpha_1. \text{ Place the triangles with this angle at the origin and the } b \text{ sides along the } x \text{ axis. Take the } b \text{ sides to be the bases of the triangles. Then the heights are } h_i = c_i \sin \alpha_i \text{ and we get} \]

\[ h_2 \geq h_1 \]

The areas are then

\[ A_2 = \frac{1}{2} b_2 h_2 \geq \frac{1}{2} b_1 h_1 = A_1. \]

A3 Define a sequence \( \{u_n\}_{n=0}^{\infty} \) by \( u_0 = u_1 = u_2 = 1 \), and thereafter by the condition that

\[ \det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n! \]

for all \( n \geq 0 \). Show that \( u_n \) is an integer for all \( n \). (By convention, \( 0! = 1 \).)

\[ A: \text{In fact for } n \geq 1 \]

\[ u_n = (n - 1)(n - 3) \cdots \]

with the product terminating at 1 if \( n - 1 \) is odd and at 2 if \( n - 1 \) is even. We will prove this and that \( u_n u_{n-1} = (n - 1)! \)

by induction on \( n \). For \( n = 1 \) and \( n = 2 \) the first formula is given. Since \( u_1 u_2 = 1 \) the product formula holds for \( n = 1 \). Now if the formulas hold for \( m \leq n + 2 \) then

\[ u_{n+3} u_n = n! + u_{n+2} u_{n+1} = n! \cdot (n + 1)! = (n + 2)! \]

It is elementary that \( n!/u_n = (n - 2)! \cdots \) and so

\[ u_{n+3} = (n + 2)(n - 2) \cdots \]

Finally check that \( u_{n+3} u_{n+2} = (n + 2)! \) as required.
A4 Show that for any positive integer \( n \), there is an integer \( N \) such that the product \( x_1 x_2 \cdots x_n \) can be expressed identically in the form

\[
x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n
\]

where the \( c_i \) are rational numbers and each \( a_{ij} \) is one of the numbers \(-1, 0, 1\).

\[ A: \text{For each nonempty subset } J \text{ of } \{1, \cdots, n\} \text{ define} \]

\[
q_J = \left( \sum_{j \in J} x_j \right)^n
\]

and let \( R \) be the set of vectors \( r \) with entries \( r_1, \ldots, r_n \) which are non-negative integers summing to \( n \). Let

\[
\binom{n}{r} = \frac{n!}{\prod r_j!}
\]

denote a multinomial coefficient. Then

\[
q_{\{1,\ldots,n\}} = (x_1 + \cdots + x_n)^n = \sum_{r \in R} \binom{n}{r} \prod x_j^{r_j}
\]

Let

\[
A_J = \{ r \in R : r_j = 0 \text{ for all } j \notin J \}
\]

and

\[
B_J = \{ r \in A_J : r_j > 0 \text{ for all } j \in J \}
\]

Define

\[
p_J = \sum_{r \in B_J} \binom{n}{r} \prod x_j^{r_j}
\]

and note that for each \( J \subset R \) we have

\[
q_J = \sum_{r \in A_J} \binom{n}{r} \prod x_j^{r_j} = \sum_{J' \subset J} p_{J'}.
\]

We now claim that we can write each \( p_J \) as a linear combination with rational coefficients \( c(J', J) \) of the form

\[
p_J = \sum_{J' \subset J} c(J', J) q_{J'}. \]

We do this by induction on the cardinality of \( J \). For \( J \) a singleton, say \( J = \{j\} \) we see that \( A_J = B_J \) and \( q_J = p_J \). Now if the result has been established for all strict subsets \( J' \) of some subset \( J \) of \( \{1, \ldots, n\} \) then

\[
q_J = p_J + \sum_{J' \subset J, J' \neq J} p_{J'} = p_J + \sum_{J' \subset J, J' \neq J} \sum_{J'' \subset J'} c(J'', J') q_{J''}.
\]

which can be solved for \( p_J \) to give the result. In particular,

\[
p_{\{1,\ldots,n\}}
\]

can be written as a linear combination of the \( q_J \) with rational coefficients as required.

\[ \blacksquare \]
A5 An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, $p$ and $q$, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at $p$ and ending at $q$, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

**A:** Let $N_{n,m}$ denote the random number of components in an $n \times m$ checkerboard coloured as described. Let $\mu_{n,m} = \mathbf{E}(N_{n,m})$ and note $\mu_{n,m} = \mu_{m,n}$. Since $N_{1,1} = 1$ we find $\mu_{1,1} = 1 \geq 1 \times 1/8$. By induction (and the symmetry noted above) it suffices to show that $\mu_{n,m} \geq mn/8$ implies $\mu_{n,m+1} \geq (n(m+1))/8$. Consider an $n \times (m+1)$ checkerboard. Let $N_{n,m}$ denote the number of distinct components in the $n \times m$ checkerboard obtained by striking off column $m+1$. We will say that an isolated single column component begins in row $i$ if there is an integer $k \geq 1$ such that

- squares $i, \ldots, i+k-1$ in column $m+1$ are the same colour.
- squares $i-1$ and $i+k$ in column $m+1$ are not the same colour as those from $i$ to $i+k-1$. (If $i = 1$ or $i+k-1 = n$ then this condition is satisfied by definition.)
- squares $i, \ldots, i+k-1$ in column $m$ are the other colour from the same numbered squares in column $m+1$.

We will say that a join begins in row $i$ if there is an integer $k \geq 2$ such that squares $(i,m), (i,m+1), (i+1,m+1), \ldots, (i+k,m+1), (i+k,m)$ are the same colour and squares $(i+1,m), \ldots, (i+k-1,m)$ are the other colour. Call $k$ in these two definitions the length of the component or the join.

Let $I_i$ take the value 1 if an isolated single column component begins in row $i$ and the value 0 otherwise. Similarly let $J_i$ be 1 or 0 according as a join starts in row $i$. The number, $N_{n,m+1}$, of components in the whole board is at least $N_{n,m} + \sum_{i=1}^n I_i - \sum_{i=1}^n J_i$.

(If is not equal because the joins sometimes actually connect 2 squares which are already connected.) I claim that $E(I_i - J_i) \geq 1/8$

If so then $\mu_{n,m+1} \geq \mu_{n,m} + n/8$

which would prove the result by induction.

It remains to establish the claim. The probability that an isolated component of length $k$ begins in row $i$ is $2^{-2k-1}$ (for $i > 1$ and $i+k < n+1$). It is $2^{-2k}$ if $i = 1$ and $i+k < n+1$ or $i > 1$ and $i+k = n+1$. It is $2^{-2k+1}$ if $i = 1$ and $i+k = n+1$. The expectation of $I_i$ is the sum over $k$ from 1 to $n+1-i$ of these probabilities. Thus

$E(I_i) \geq \sum_{k=1}^{n+1-i} 2^{-2k-1}$

On the other hand the probability of a join of length $k$ beginning in row $i$ is $2^{-2k-1}$ for all $i$ and all $k \geq 2$ such that $i+k \leq n$. Thus

$E(J_i) = \sum_{k=2}^{n-i} 2^{-2k-1}$

Hence

$E(I_i - J_i) \geq 2^{-3} = 1/8$

as required.
A6 Suppose that \( f(x, y) \) is a continuous real-valued function on the unit square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \). Show that

\[
\int_0^1 \left( \int_0^1 f(x,y) \, dx \right)^2 \, dy + \int_0^1 \left( \int_0^1 f(x,y) \, dy \right)^2 \, dx \\
\leq \left( \int_0^1 \int_0^1 f(x,y) \, dx \, dy \right)^2 + \int_0^1 \left( \int_0^1 f(x,y) \right)^2 \, dx \, dy.
\]

A: Put \( H(x) = \int_0^1 f(x,y) \, dy \) and \( G(y) = \int_0^1 f(x,y) \, dx \). Put

\[
c = \int_0^1 \int_0^1 f(x,y) \, dx \, dy = \int_0^1 H(x) \, dx = \int_0^1 G(y) \, dy.
\]

Then

\[
0 \leq \int_0^1 \int_0^1 \{ f(x,y) - H(x) - G(y) + c \}^2 \, dx \, dy \\
= \int_0^1 \int_0^1 \{ f^2(x,y) + H^2(x) + G^2(y) + c^2 - 2(f(x,y)H(x) + f(x,y)G(y) - cf(x,y) + cH(x) + cG(y) - H(x)G(y)) \} \, dx \, dy \\
= \int_0^1 \int_0^1 f^2(x,y) \, dx \, dy + \int_0^1 H^2(x) \, dx + \int_0^1 G^2(y) \, dy + c^2 - 2 \int_0^1 H^2(x) \, dx - 2 \int_0^1 G^2(y) \, dy + 2c^2 - 2c^2 - 2c^2 + 2c^2 \\
= \int_0^1 \int_0^1 f^2(x,y) \, dx \, dy - \int_0^1 H^2(x) \, dx - \int_0^1 G^2(y) \, dy + c^2.
\]

Rearranging gives

\[
\int_0^1 H^2(x) \, dx + \int_0^1 G^2(y) \, dy \leq c^2 + \int_0^1 \int_0^1 f^2(x,y) \, dx \, dy
\]

as desired.
B1 Let $P(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that $r$ is a rational number such that $P(r) = 0$. Show that the $n$ numbers
\[
c_n r, \ c_n r^2 + c_{n-1} r, \ c_n r^3 + c_{n-1} r^2 + c_{n-2} r,
\ldots, \ c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r
\]
are integers.

A: For $n = 1$ the result is obvious. Since $P(r) = 0$ we see that
\[
c_n r^n + \cdots + c_1 r = -c_0
\]
is an integer. Suppose now that the result has been established for all polynomials of degree less than $n$. We claim that there is an integer $d$ such that
\[
c_n r^{n-1} + c_{n-1} r^{n-2} + \cdots + c_2 r + d = 0.
\]
If so then let
\[
P^*(x) = c_n x^{n-1} + c_{n-1} x^{n-2} + \cdots + c_2 x + d.
\]
Since $P^*(r) = 0$ we see by the induction hypothesis that
\[
c_n r, c_n r^2 + c_{n-1} r, c_n r^3 + c_{n-1} r^2 + c_{n-2} r,
\ldots, c_n r^{n-1} + c_{n-1} r^{n-2} + \cdots + c_2 r
\]
are integers which would finish the problem.

It remains to find $d$. There is no loss in assuming that $r = p/q$ for integers $p$ and $q$ which are relatively prime. Moreover there is no loss in assuming that the greatest common divisor of the integers $c_n, \ldots, c_0$ is 1. Multiply $P(r) = 0$ by $q^n$ and see
\[
c_n p^n + c_{n-1} p^{n-1} q + \cdots + c_0 q^n = 0.
\]
This shows that $c_0$ is divisible by $p$; say $c_0 = pc_0^*$ for some integer $c_0^*$. We see that
\[
c_n p(p^{n-1}) + c_{n-1} pq(p^{n-2}) + \cdots + c_1 pq^{n-1} + c_0^* pq^n = 0
\]
Divide through by $pq^{n-1}$ to see
\[
c_0^* qr^{n-1} + c_{n-1} r^{n-2} + \cdots + c_1 + c_0^* q = 0.
\]
This gives $d = c_1 + c_0^* q$ which is clearly an integer finishing the proof.

B2 Let $m$ and $n$ be positive integers. Show that
\[
\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.
\]

A: In fact the probability of $m$ successes in $m+n$ independent Bernoulli trials with probability $p$ of success on an individual trial is
\[
\frac{(m+n)!}{m!n!} p^m (1-p)^n < 1
\]
for all $p$ not 0 or 1 and all positive integers $m$ and $n$. In particular the inequality holds at $p = m/(m+n)$ giving
\[
\frac{(m+n)!}{m!n!} \left( \frac{m}{m+n} \right)^m \left( \frac{n}{m+n} \right)^n < 1
\]
Multiply through by $(m+n)^{m+n} m!n!$ to get the desired inequality.
B3 Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y); 0 \leq x \leq a, 0 \leq y \leq f(x)\}$$

has perimeter $k$ units and area $k$ square units for some real number $k$.

A: There is such a function if and only if $a > 2$. First consider the function $f(x) \equiv c$. The integral is $ca$ and the perimeter is $2a + 2c$. Set

$$ca = 2c + 2a$$

and solve for $c$ to find $c = 2a/(a - 2)$ which is positive for $a > 2$. On the other hand for a general function $f$ let $c = \max\{f(x)0 \leq x \leq a\}$. Then

$$\text{Area} \leq ac \leq 2c < 2c + a \leq \text{Perimeter}$$

At the same time the perimeter is at least the length $a$ of the base of the figure plus $2c$ since the figure must get from $(0,0)$ up to some point $(x,c)$ and then back down to $(a,0)$. For $0 < a \leq 2$ we then get

$$\text{Area} \leq ac \leq 2c$$

$< 2c + a$ 

$\leq \text{Perimeter}$
Let \( n \) be a positive integer, \( n \geq 2 \), and put \( \theta = 2\pi / n \). Define points \( P_k = (k, 0) \) in the \( xy \)-plane, for \( k = 1, 2, \ldots, n \). Let \( R_k \) be the map that rotates the plane counterclockwise by the angle \( \theta \) about the point \( P_k \). Let \( R \) denote the map obtained by applying, in order, \( R_1 \), then \( R_2 \), . . . , then \( R_n \). For an arbitrary point \((x, y)\), find, and simplify, the coordinates of \( R(x, y) \).

**A:** We will use matrix representations to do this problem. The matrix

\[
R = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

rotates a vector \( \mathbf{v} \) counterclockwise by the angle \( \theta \) about the origin. Let \( \mathbf{x}_1 \) be some fixed vector in \( \mathbb{R}^2 \); in the question this will be

\[
\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

Let

\[
\mathbf{u}_0 = \begin{bmatrix} x \\ y \end{bmatrix}
\]

denote the initial point and then let \( \mathbf{u}_{k+1} \) be \( \mathbf{u}_k \) rotated clockwise by \( \theta \) around the point \( (k+1)\mathbf{x}_1 \). The question asks for a simple formula for \( \mathbf{u}_n \) when \( \theta = 2\pi / n \). Note that

\[
\mathbf{u}_{k+1} = R \{ \mathbf{u}_k - (k+1)\mathbf{x}_1 \} + (k+1)\mathbf{x}_1.
\]

Define \( \mathbf{v}_k = \mathbf{u}_k - k\mathbf{x}_1 \) and see that

\[
\mathbf{v}_{k+1} = R\mathbf{v}_k - \mathbf{a}
\]

where \( \mathbf{a} \) is shorthand for \( R\mathbf{x}_1 \). Put \( \mathbf{b} = - (I - R)^{-1} \mathbf{a} \) where \( I \) is the \( 2 \times 2 \) identity matrix. Then

\[
\mathbf{v}_{k+1} - \mathbf{b} = R (\mathbf{v}_k - \mathbf{b})
\]

It follows inductively that

\[
\mathbf{v}_n - \mathbf{b} = R^n (\mathbf{v}_0 - \mathbf{b})
\]

Note that \( R^k \) rotates by \( k\theta \). For \( k = n \) and \( \theta = 2\pi / n \) we see that \( R^n \) rotates by the angle \( 2\pi \) which means that \( R^n \) is simply \( I \).

Thus

\[
\mathbf{v}_n = \mathbf{v}_0
\]

and so

\[
\mathbf{u}_n = n\mathbf{x}_1 + \mathbf{u}_0 = \begin{bmatrix} n + x \\ y \end{bmatrix}
\]
B5 Evaluate

\[
\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left( \frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.
\]

A: It suffices, by taking logs, to compute

\[
a = \lim_{x \to 1^-} \sum_{n=0}^{\infty} x^n \log \left( \frac{1 + x^{n+1}}{1 + x^n} \right).
\]

Write \(\log(1 + y) = y + r(y)\) and note that there is a \(C\) such that

\[
|r(y)| \leq cy^2
\]

for all \(|y| < 1/2\). Write \(y_n(x) = (x - 1)x^n/(1 + x^n)\). Then for \(x > 1/2\) we find \(|y_n(x)| \leq 1/2\) for all \(n\). Hence

\[
\sum_{n=0}^{\infty} x^n \left( \log \left( \frac{1 + x^{n+1}}{1 + x^n} \right) - y_n(x) \right) \leq C(x - 1)^2 \sum_{n=0}^{\infty} x^{3n}/(1 + x^n)^2
\]

\[
\leq C(x - 1)^2 \sum_{n=0}^{\infty} x^{3n}
\]

\[
= C \frac{(1 - x)^2}{1 - x^3}
\]

\[
= C \frac{1 - x}{1 + x + x^2}
\]

which tends to 0 as \(x \to 1\). Hence

\[
a = \lim_{x \to 1^-} (x - 1) \sum_{n=0}^{\infty} x^{2n}/(1 + x^n)
\]

Now write \(x = e^{-\delta}\) to find

\[
a = \lim_{\delta \to 0^+} \frac{(e^{-\delta} - 1)}{\delta} \sum_{n=0}^{\infty} e^{-2n\delta}/(1 + e^{-n\delta})
\]

The term

\[
\frac{(e^{-\delta} - 1)}{\delta} \to -1
\]

as \(\delta \to 0\) (definition of derivative!). The sum above is a Riemann sum for the integral

\[
I = \int_0^{\infty} \frac{e^{-2x}}{1 + e^{-x}} \, dx
\]

Since the function

\[
f(x) = \frac{e^{-2x}}{1 + e^{-x}}
\]

is easily seen to be monotone decreasing we see by comparison that

\[
\delta \sum_{n=1}^{\infty} e^{-2n\delta}/(1 + e^{-n\delta}) \leq I \leq \delta \sum_{n=0}^{\infty} e^{-2n\delta}/(1 + e^{-n\delta})
\]

Since the right hand side and left hand side differ by \(\delta/2 \to 0\) we see that \(a = I\). It remains to compute \(I\). Substitute \(u = e^{-x}\) and \(du = e^{-x} \, dx\) to find

\[
I = \int_0^1 \frac{u}{1 + u} \, du = 1 - \log(2)
\]

Thus the desired limit is

\[
\exp(-a) = \frac{2}{e}.
\]
Let \( A \) be a non-empty set of positive integers, and let \( N(x) \) denote the number of elements of \( A \) not exceeding \( x \). Let \( B \) denote the set of positive integers \( b \) that can be written in the form \( b = a - a' \) with \( a, a' \in A \) and \( a' < a \). Let \( b_1 < b_2 < \cdots \) be the members of \( B \), listed in increasing order. Show that if the sequence \( b_{i+1} - b_i \) is unbounded, then

\[
\lim_{x \to \infty} \frac{N(x)}{x} = 0.
\]

**A**: It will be convenient to let \( N(A) \) denote the number of members of \( A \) in \( A \). We will show first that we can choose a sequence

\[
k_1 < k_2 < \cdots
\]

of integers in such a way that any interval of \( k_n \) consecutive integers intersects \( A \) in at most \( 2^{-n} k_n \) points. Take \( m_1 \) to be some integer not in \( B \). For any \( x \) break the set \( \{ x + 1, x + 2, \ldots, x + 2m_1 \} \) into \( m_1 \) pairs of integers \( y, y + m_1 \). In each pair at most one member can belong to \( A \). It follows that the number of integers in \( \{ x + 1, x + 2, \ldots, x + 2m_1 \} \) belonging to \( A \) is at most \( m_1 \). Thus we may put \( k_1 = 2m_1 \).

Now suppose, for an inductive proof that \( k_1, \ldots, k_n \) have already been found with the desired properties. There is an \( i \) such that \( b_{i+1} - b_i > 4k_n \). Then there is an integer \( m \) such that

\[
b_i < (m - 1)k_n + 1 < (m + 1)k_n - 1 < b_{i+1}.
\]

Put

\[
k_{n+1} = 2mk_n.
\]

Fix any integer \( x \geq 0 \) and consider the sequence \( x + 1, x + 2, \ldots, x + k_{n+1} \). Let

\[
L_1 = \{x + 1, \ldots, x + k_n\}
\]

\[
L_2 = \{x + k_n + 1, \ldots, x + 2k_n\}
\]

\[
\vdots
\]

\[
L_m = \{x + (m - 1)k_n + 1, \ldots, x + mk_n\}
\]

\[
R_1 = \{x + mk_n + 1, \ldots, x + (m + 1)k_n\}
\]

\[
\vdots
\]

\[
R_m = \{x + (2m - 1)k_n + 1, \ldots, x + 2mk_n\}
\]

Notice that if \( y' \in R_j \) and \( y \in L_j \) then

\[
(m - 1)k_n + 1 \leq y' - y \leq (m + 1)k_n - 1
\]

so that \( y' - y \notin B \). Hence if \( L_j \cap A \) is not empty then \( R_j \cap A \) must be empty and vice versa. Since each of \( L_j \) and \( R_j \) has length \( k_n \), we find

\[
N(L_j \cup R_j) = \max\{N(L_j), N(R_j)\} \leq 2^{-n} k_n
\]

by our induction assumption. Taking the union for \( j \) from 1 to \( m \) we find

\[
N(x + k_{n+1}) - N(x) \leq 2^{-n} mk_n = 2^{-(n+1)} k_{n+1}.
\]

Now suppose \( x \) is any integer larger than \( k_n \). Then there is some positive integer \( l \) such that \(lk_n \leq x < (l+1)k_n\). Then

\[
\frac{N(x)}{x} \leq \frac{N((l+1)k_n)}{lk_n} \leq \frac{l+1}{l} 2^{-n} \leq 2^{-(n-1)}.
\]

It follows that

\[
\lim_{x \to \infty} \frac{N(x)}{x} = 0.
\]