The 65th William Lowell Putnam Mathematical Competition
Saturday, December 4, 2004

A1 Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

A: *Yes. Let $S(n)$ be the number of successes in the first n throws. Then $F(n) = n - S(n)$ is the number of failures. Put*

$$W(n) = S(n) - 4F(n)$$

and note tha the success percentage is less than 80, exactly 80 or more than 80 according as $W(n)$ is negative, 0 or positive. Notice too that either $W(n+1) = W(n) + 1$ or $W(n+1) = W(n) - 4$. Let $M \leq N$ be the least $n > 1$ such that $W(n) > 0$. Such an M exists from the assumptions. Now $W(M-1) \leq 0$ by definition of M and so

$$W(M) \leq W(M-1) + 1 \leq 1$$

Since $W(M)$ is an integer and $W(M) > 0$ we find $W(M) = 1$. This shows

$$W(M-1) \geq W(M) - 1 \geq 0$$

and so $W(M-1) = 0$. That is, at toss $M-1$ the success rate was exactly $4/5$. •

A2 For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$?

A: *Yes. Let $\alpha_i, \beta_i, \gamma_i$ be the angles opposite sides a_i, b_i, c_i respectively. Since the two sets of angles have the same sum there is an angle on triangle 2 which is larger than the corresponding angle on triangle 1. Without loss suppose $\alpha_2 \geq \alpha_1$. Place the triangles with this angle at the origin and the b sides along the x axis. Take the b sides to be the bases of the triangles. Then the heights are $h_i = c_i \sin \alpha_i$ and we get*

$$h_2 \geq h_1$$

The areas are then

$$A_2 = \frac{1}{2} b_2 h_2 \geq \frac{1}{2} b_1 h_1 = A_1.$$

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A3 Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

A: *In fact for $n \geq 1$*

$$u_n = (n-1)(n-3)\cdots$$

with the product terminating at 1 if $n-1$ is odd and at 2 if $n-1$ is even. We will prove this and that $u_n u_{n-1} = (n-1)!$ by induction on n. For $n = 1$ and $n = 2$ the first formula is given. Since $u_1 u_2 = 1 = 1!$ the product formula holds for $n = 1$. Now if the formulas hold for $m \leq n+2$ then

$$u_{n+3} u_n = n! + u_{n+2} u_{n+1} = n! + (n+1)! = (n+2)n!$$

It is elementary that $n!/u_n = n(n-2)\cdots$ and so

$$u_{n+3} = (n+2)n(n-2)\cdots$$

Finally check that $u_{n+3} u_{n+2} = (n+2)!$ as required. •

A4 Show that for any positive integer n, there is an integer N such that the product $x_1x_2\cdots x_n$ can be expressed identically in the form

$$x_1x_2\cdots x_n = \sum_{i=1}^{N} c_i(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers $-1, 0, 1$.

A: *For each nonempty subset J of $\{1,\cdots,n\}$ define*

$$q_J = \left(\sum_{j\in J} x_j\right)^n$$

and let $\mathcal{R}$ be the set of vectors $\mathbf{r}$ *with entries $r_1,\ldots,r_n$ which are non-negative integers summing to n. Let*

$$\binom{n}{\mathbf{r}} = \left(\frac{n!}{\prod r_j!}\right)$$

denote a multinomial coefficient. Then

$$q_{\{1,\cdots,n\}} = (x_1 + \cdots + x_n)^n = \sum_{\mathbf{r}\in\mathcal{R}} \binom{n}{\mathbf{r}} \prod x_j^{r_j}$$

Let

$$A_J = \{\mathbf{r}\ in \mathcal{R} : r_j = 0 \textit{ for all } j \notin J\}$$

and

$$B_J = \{\mathbf{r}\ in A_J : r_j > 0 \textit{ for all } j \in J\}$$

Define

$$p_J = \sum_{\mathbf{r}\in B_J} \binom{n}{\mathbf{r}} \prod x_j^{r_j}$$

and note that for each $J \subset \mathcal{R}$ we have

$$q_J = \sum_{\mathbf{r}\in A_J} \binom{n}{\mathbf{r}} \prod x_j^{r_j} = \sum_{J'\subset J} p_{J'}.$$

We now claim that we can write each p_J as a linear combination with rational coefficients $c(J', J)$ of the form

$$p_J = \sum_{J'\subset J} c(J', J) q_{J'}.$$

We do this by induction on the cardinality of J. For J a singleton, say $J = \{j\}$ we see that $A_J = B_J$ and $q_J = p_J$. Now if the result has been established for all strict subsets J' of some subset J of $\{1,\ldots,n\}$ then

$$q_J = p_J + \sum_{J'\subset J, J'\neq J} p_{J'} = p_J + \sum_{J'\subset J, J'\neq J}\ \sum_{J''\subset J'} c(J'', J') q_{J''}.$$

which can be solved for p_J to give the result. In particular,

$$p_{1,\ldots,n}$$

can be written as a linear combination of the q_J with rational coefficients as required. •

A5 An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

A: *Let $N_{n,m}$ denote the random number of components in an $n \times m$ checkerboard coloured as described. Let $\mu_{n,m} = \mathrm{E}(N_{n,m})$ and note $\mu_{n,m} = \mu_{m,n}$. Since $N_{1,1} = 1$ we find $\mu_{1,1} = 1 \geq 1 \times 1/8$. By induction (and the symmetry noted above) it suffices to show that $\mu_{n,m} \geq nm/8$ implies $\mu_{n,m+1} \geq n(m+1)/8$. Consider an $n \times (m+1)$ checkerboard. Let $N_{n,m}$ denote the number of distinct components in the $n \times m$ checkerboard obtained by striking off column $m+1$.*

We will say that an isolated single column component begins in row i if there is an integer $k \geq 1$ such that

- *squares $i, \ldots, i+k-1$ in column $m+1$ are the same colour.*
- *squares $i-1$ and $i+k$ in column $m+1$ are not the same colour as those from i to $i+k-1$. (If $i = 1$ or $i+k-1 = n$ then this condition is satisfied by definition.)*
- *squares $i, \ldots, i+k-1$ in column m are the other colour from the same numbered squares in column $m+1$.*

We will say that a join begins in row i if there is an integer $k \geq 2$ such that squares $(i,m), (i,m+1), (i+1,m+1), \ldots, (i+k,m+1), (i+k,m)$ are the same colour and squares $(i+1,m), \ldots, (i+k-1,m)$ are the other colour. Call k in these two definitions the length of the component or the join.

Let I_i take the value 1 if an isolated single column component begins in row i and the value 0 otherwise. Similarly let J_i be 1 or 0 according as a join starts in row i. The number, $N_{n,m+1}$, of components in the whole board is at least

$$N_{n,m} + \sum_{i=1}^{n} I_i - \sum_{i=1}^{n} J_i$$

(It is not equal because the joins sometimes actually connect 2 squares which are already connected.) I claim that

$$E(I_i - J_i) \geq 1/8$$

If so then

$$\mu_{n,m+1} \geq \mu_{n,m} + n/8$$

which would prove the result by induction.

It remains to establish the claim. The probability that an isolated component of length k begins in row i is 2^{-2k-1} (for $i > 1$ and $i+k < n+1$). It is 2^{-2k} if $i = 1$ and $i+k < n+1$ or $i > 1$ and $i+k = n+1$. It is 2^{-2k+1} if $i = 1$ and $i+k = n+1$. The expectation of I_i is the sum over k from 1 to $n+1-i$ of these probabilities. Thus

$$\mathrm{E}(I_i) \geq \sum_{k=1}^{n+1-i} 2^{-2k-1}$$

On the other hand the probability of a join of length k beginning in row i is 2^{-2k-1} for all i and all $k \geq 2$ such that $i+k \leq n$. Thus

$$\mathrm{E}(J_i) = \sum_{k=2}^{n-i} 2^{-2k-1}$$

Hence

$$\mathrm{E}(I_i - J_i) \geq 2^{-3} = 1/8$$

as required. •

A6 Suppose that $f(x,y)$ is a continuous real-valued function on the unit square $0 \le x \le 1, 0 \le y \le 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y)dx\right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy\right)^2 dx$$
$$\le \left(\int_0^1 \int_0^1 f(x,y)dx\,dy\right)^2 + \int_0^1 \int_0^1 (f(x,y))^2\,dx\,dy.$$

A: *Put* $H(x) = \int_0^1 f(x,y)dy$ *and* $G(y) = \int_0^1 f(x,y)dx$. *Put*

$$c = \int_0^1 \int_0^1 f(x,y)dxdy = \int_0^1 H(x)dx = \int_0^1 G(y)dy.$$

Then

$$\begin{aligned} 0 &\le \int_0^1 \int_0^1 \{f(x,y) - H(x) - G(y) + c\}^2\,dxdy \\ &= \int_0^1 \int_0^1 \{f^2(x,y) + H^2(x) + G^2(y) + c^2 \\ &\quad -2\,(f(x,y)H(x) + f(x,y)G(y) - cf(x,y) + cH(x) + cG(y) - H(x)G(y))\}\,dxdy \\ &= \int_0^1 \int_0^1 f^2(x,y)dxdy + \int_0^1 H^2(x)dx + \int_0^1 G^2(y)dy + c^2 \\ &\quad -2\int_0^1 H^2(x)dx - 2\int_0^1 G^2(y)dy + 2c^2 - 2c^2 - 2c^2 + 2c^2 \\ &= \int_0^1 \int_0^1 f^2(x,y)dxdy - \int_0^1 H^2(x)dx - \int_0^1 G^2(y)dy + c^2. \end{aligned}$$

Rearranging gives

$$\int_0^1 H^2(x)dx + \int_0^1 G^2(y)dy \le c^2 + \int_0^1 \int_0^1 f^2(x,y)dxdy$$

as desired. •

B1 Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_n r,\ c_n r^2 + c_{n-1} r,\ c_n r^3 + c_{n-1} r^2 + c_{n-2} r,$$
$$\ldots,\ c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

A: *For $n = 1$ the result is obvious. Since $P(r) = 0$ we see that*

$$c_n r^n + \cdots + c_1 r = -c_0$$

is an integer. Suppose now that the result has been established for all polynomials of degree less than n. We claim that there is an integer d such that

$$c_n r^{n-1} + c_{n-1} r^{n-2} + \cdots + c_2 r + d = 0.$$

If so then let

$$P^*(x) = c_n x^{n-1} + c_{n-1} x^{n-2} + \cdots + c_2 x + d.$$

Since $P^*(r) = 0$ we see by the induction hypothesis that

$$c_n r,\ c_n r^2 + c_{n-1} r,\ c_n r^3 + c_{n-1} r^2 + c_{n-2} r,$$
$$\ldots,\ c_n r^{n-1} + c_{n-1} r^{n-2} + \cdots + c_2 r$$

are integers which would finish the problem.

It remains to find d. There is no loss in assuming that $r = p/q$ for integers p and q which are relatively prime. Moreover there is no loss in assuming that the greatest common divisor of the integers $c_n, \ldots, c_0$ is 1. Multiply $P(r) = 0$ by q^n and see

$$c_n p^n + c_{n-1} p^{n-1} q + \cdots + c_0 q^n = 0.$$

This shows that c_0 is divisible by p; say $c_0 = p c_0^*$ for some integer c_0^*. We see that

$$c_n p(p^{n-1}) + c_{n-1} pq(p^{n-2}) + \cdots c_1 pq^{n-1} + c_0^* pq^n = 0$$

Divide through by pq^{n-1} to see

$$c_n^* q r^{n-1} + c_{n-1} r^{n-2} + \cdots c_1 + c_0^* q = 0.$$

This gives $d = c_1 + c_0^* q$ which is clearly an integer finishing the proof. •

B2 Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

A: *In fact the probability of m successes in $m + n$ independent Bernoulli trials with probability p of success on an individual trial is*

$$\frac{(m+n)!}{m!n!} p^m (1-p)^n < 1$$

for all p not 0 or 1 and all positive integers m and n. In particular the inequality holds at $p = m/(m+n)$ giving

$$\frac{(m+n)!}{m!n!} \left(\frac{m}{m+n}\right)^m \left(\frac{n}{m+n}\right)^n < 1$$

Multiply through by $(m+n)^{m+n} m!n!$ to get the desired inequality. •

B3 Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y); 0 \leq x \leq a, 0 \leq y \leq f(x)\}$$

has perimeter k units and area k square units for some real number k.

A: *There is such a function if and only if $a > 2$. First consider the function $f(x) \equiv c$. The integral is ca and the perimeter is $2a + 2c$. Set*

$$ca = 2c + 2a$$

and solve for c to find $c = 2a/(a - 2)$ which is positive for $a > 2$. On the other hand for a general function f let $c = \max\{f(x) 0 \leq x \leq a\}$. Then

$$\textit{Area} \leq ac$$

At the same time the perimeter is at least the length a of the base of the figure plus $2c$ since the figure must get from $(0, 0)$ up to some point (x, c) and then back down to $(a, 0)$. For $0 < a \leq 2$ we then get

$$\begin{aligned} \textit{Area} &\leq ac \\ &\leq 2c \\ &< 2c + a \\ &\leq \textit{Perimeter} \end{aligned}$$

•

B4 Let n be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, 2, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then $R_2, \ldots$, then R_n. For an arbitrary point (x, y), find, and simplify, the coordinates of $R(x, y)$.

A: *We will use matrix representations to do this problem. The matrix*

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

rotates a vector $\mathbf{v}$ *counterclockwise by the angle* θ *about the origin. Let* $\mathbf{x}_1$ *be some fixed vector in* R^2*; in the question this will be*

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let

$$\mathbf{u}_0 = \begin{bmatrix} x \\ y \end{bmatrix}$$

denote the initial point and then let $\mathbf{u}_{k+1}$ *be* $\mathbf{u}_k$ *rotated clockwise by* θ *around the point* $(k+1)\mathbf{x}_1$*. The question asks for a simple formula for* $\mathbf{u}_n$ *when* $\theta = 2\pi/n$*. Note that*

$$\mathbf{u}_{k+1} = \mathbf{R}\left\{\mathbf{u}_k - (k+1)\mathbf{x}_1\right\} + (k+1)\mathbf{x}_1.$$

Define $\mathbf{v}_k = \mathbf{u}_k - k\mathbf{x}_1$ *and see that*

$$\mathbf{v}_{k+1} = \mathbf{R}\mathbf{v}_k - \mathbf{a}$$

where $\mathbf{a}$ *is shorthand for* $\mathbf{R}\mathbf{x}_1$*. Put* $\mathbf{b} = -\left(\mathbf{I} - \mathbf{R}\right)^{-1}\mathbf{a}$ *where* $\mathbf{I}$ *is the* 2×2 *identity matrix. Then*

$$\mathbf{v}_{k+1} - \mathbf{b} = \mathbf{R}\left(\mathbf{v}_k - \mathbf{b}\right)$$

It follows inductively that

$$\mathbf{v}_n - \mathbf{b} = \mathbf{R}^n\left(\mathbf{v}_0 - \mathbf{b}\right)$$

Note that $\mathbf{R}^k$ *rotates by* $k\theta$*. For* $k = n$ *and* $\theta = 2\pi/n$ *we see that* $\mathbf{R}^n$ *rotates by the angle* 2π *which means that* $\mathbf{R}^n$ *is simply* $\mathbf{I}$*.*

Thus

$$\mathbf{v}_n = \mathbf{v}_0$$

and so

$$\mathbf{u}_n = n\mathbf{x}_1 + \mathbf{u}_0 = \begin{bmatrix} n + x \\ y \end{bmatrix}$$

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B5 Evaluate

$$\lim_{x\to 1^-}\prod_{n=0}^{\infty}\left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}.$$

A: *It suffices, by taking logs, to compute*

$$a \equiv \lim_{x\to 1^-}\sum_{n=0}^{\infty} x^n \log\left(\frac{1+x^{n+1}}{1+x^n}\right).$$

Write $\log(1+y) = y + r(y)$ *and note that there is a* C *such that*

$$|r(y)| \le cy^2$$

for all $|y| < 1/2$. *Write* $y_n(x) = (x-1)x^n/(1+x^n)$. *Then for* $x > 1/2$ *we find* $|y_n(x)| \le 1/2$ *for all* n. *Hence*

$$\begin{aligned}\sum_{n=0}^{\infty}\left|x^n\left\{\log\left(\frac{1+x^{n+1}}{1+x^n}\right) - y_n(x)\right\}\right| &\le C(x-1)^2\sum_{n=0}^{\infty} x^{3n}/(1+x^n)^2\\ &\le C(x-1)^2\sum_{n=0}^{\infty} x^{3n}\\ &= C\frac{(1-x)^2}{1-x^3}\\ &= C\frac{1-x}{1+x+x^2}\end{aligned}$$

which tends to 0 as $x \to 1$. *Hence*

$$a = \lim_{x\to 1-}(x-1)\sum_{n=0}^{\infty} x^{2n}/(1+x^n)$$

Now write $x = e^{-\delta}$ *to find*

$$a = \lim_{\delta\to 0+}\frac{(e^{-\delta}-1)}{\delta}\delta\sum_{n=0}^{\infty} e^{-2n\delta}/(1+e^{-n\delta})$$

The term

$$\frac{(e^{-\delta}-1)}{\delta} \to -1$$

as $\delta \to 0$ *(definition of derivative!). The sum above is a Riemann sum for the integral*

$$I = \int_0^{\infty}\frac{e^{-2x}}{1+e^{-x}}dx$$

Since the function

$$f(x) = \frac{e^{-2x}}{1+e^{-x}}$$

is easily seen to be monotone decreasing we see by comparison that

$$\delta\sum_{n=1}^{\infty} e^{-2n\delta}/(1+e^{-n\delta}) \le I \le \delta\sum_{n=0}^{\infty} e^{-2n\delta}/(1+e^{-n\delta})$$

Since the right hand side and left hand side differ by $\delta/2 \to 0$ *we see that* $a = I$. *It remains to compute* I. *Substitute* $u = e^{-x}$ *and* $du = e^{-x}dx$ *to find*

$$I = \int_0^1 \frac{u}{1+u}du = 1 - \log(2)$$

Thus the desired limit is

$$\exp(-a) = \frac{2}{e}.$$

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B6 Let $\mathcal{A}$ be a non-empty set of positive integers, and let $N(x)$ denote the number of elements of $\mathcal{A}$ not exceeding x. Let $\mathcal{B}$ denote the set of positive integers b that can be written in the form $b = a - a'$ with $a \in \mathcal{A}$ and $a' \in \mathcal{A}$. Let $b_1 < b_2 < \cdots$ be the members of $\mathcal{B}$, listed in increasing order. Show that if the sequence $b_{i+1} - b_i$ is unbounded, then

$$\lim_{x\to\infty} N(x)/x = 0.$$

A: *It will be convenient to let $N(A)$ denote the number of members of $\mathcal{A}$ in A. We will show first that we can choose a sequence*

$$k_1 < k_2 < \cdots$$

of integers in such a way that any interval of k_n consecutive integers intersects $\mathcal{A}$ in at most $2^{-n}k_n$ points. Take m_1 to be some integer not in $\mathcal{B}$. For any x break the set $\{x+1, x+2, \ldots, x+2m_1\}$ into m_1 pairs of integers $y, y+m_1$. In each pair at most 1 member can belong to $\mathcal{A}$. It follows that the number of integers in $\{x+1, x+2, \ldots, x+2m_1\}$ belonging to in $\mathcal{A}$ is at most m_1. Thus we may put $k_1 = 2m_1$.

Now suppose, for an inductive proof that $k_1, \ldots, k_n$ have already been found with the desired properties. There is an i such that $b_{i+1} - b_i > 4k_n$. Then there is an integer m such than

$$b_i < (m-1)k_n + 1 < (m+1)k_n - 1 < b_{i+1}.$$

Put

$$k_{n+1} = 2mk_n.$$

Fix any integer $x \geq 0$ and consider the sequence $x+1, x+2, \ldots, x+k_{n+1}$. Let

$$\begin{aligned}
L_1 &= \{x+1, \ldots, x+k_n\} \\
L_2 &= \{x+k_n+1, \ldots, x+2k_n\} \\
&\vdots \\
L_m &= \{x+(m-1)k_n+1, \ldots, x+mk_n\} \\
R_1 &= \{x+mk_n+1, \ldots, x+(m+1)k_n\} \\
&\vdots \\
R_m &= \{x+(2m-1)k_n+1, \ldots, x+2mk_n\}
\end{aligned}$$

Notice that if $y' \in R_j$ and $y \in L_j$ then

$$(m-1)k_n + 1 \leq y' - y \leq (m+1)k_n - 1$$

so that $y' - y \notin \mathcal{B}$. Hence if $L_j \cap \mathcal{A}$ is not empty then $R_j \cap \mathcal{A}$ must be empty and vice versa. Since each of L_j and R_j has length k_n we find

$$N(L_j \cup R_j) = \max\{N(L_j), N(R_j)\} \leq 2^{-n}k_n$$

by our induction assumption. Taking the union for j from 1 to m we find

$$N(x + k_{n+1}) - N(x) \leq 2^{-n}mk_n = 2^{-(n+1)}k_{n+1}.$$

Now suppose x is any integer larger than k_n. Then there is some positive integer l such that $lk_n \leq x < (l+1)k_n$. Then

$$\frac{N(x)}{x} \leq \frac{N((l+1)k_n)}{lk_n} \leq \frac{l+1}{l}2^{-n} \leq 2^{-(n-1)}.$$

It follows that

$$lim_{x\to\infty}\frac{N(x)}{x} = 0.$$