The Forty-Seventh Annual William Lowell Putnam Competition
Saturday, December 6, 1986

Done: All

A–1 Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

Solution: *Write the constraint as*

$$(x^2 - 13/2)^2 + 36 \leq 169/4$$

or

$$(x^2 - 13/2)^2 \leq 25/4.$$

This becomes

$$|x^2 - 13/2| \leq 5/2$$

or

$$4 \leq x^2 \leq 9$$

which is the union of $-3 \leq x \leq -2$ *and* $2 \leq x \leq 3$. *The given cubic has derivative* $3(x^2 - 1)$ *which is positive for* $x > 1$ *and for* $x < -1$. *Thus the maximum over each of the two intervals occurs at the right hand end of that interval; that is, the maximum is either at* $x = -2$ *or at* $x = 3$. *The former gives the value* $-8 + 6 = -2$ *while the latter gives* $27 - 9 = 18$. *Thus the maximum value is 18 which occurs when* $x = 3$.

A–2 What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100} + 3} \right\rfloor ?$$

Solution: *Write the quantity inside the floor signs as*

$$10^m \sum_{n=0}^{\infty} \left(\frac{-3}{10^{100}} \right)^n$$

where $m = 19900$. *Terms with* $n < 199$ *give, after multiplication by* 10^m,

$$(-1)^n 3^n 10^{m-100n}$$

which is divisible by 10; such terms contribute a 0 in the units place. The terms with $n > 199$ *add up to*

$$10^m \left(\frac{3}{10^{100}} \right)^{200} \frac{1}{1 + 3/10^{100}} = \frac{3^{200}}{10^{100}(1 - 3/10^{100})}$$

which is less than 1. Finally the term with $n = 199$ *gives*

$$-3^{199}$$

so the units digit desired is just the residue class, modulo 10, of this term. In fact

$$3^4 \equiv 1 \mod 10$$

so that -3^{199} *is congruent to* -3^3 *which is -27 which is congruent to 3. So the answer is 3.*

A–3 Evaluate $\sum_{n=0}^{\infty} \mathrm{Arccot}(n^2 + n + 1)$, where $\mathrm{Arccot}\, t$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot \theta = t$.

Solution: *Suppose* $\tan\theta_n = 1/n$. *Then*

$$\tan(\theta_n - \theta_{n+1}) = \frac{\tan(\theta_n) - \tan(\theta_{n+1})}{1 + \tan(\theta_{n+1})\tan(\theta_n)} = \frac{1}{n^2+n+1}.$$

We have $\cot\phi_n = n^2+n+1$ *if and only if* $\tan\phi_n = 1/(n^2+n+1)$ *so*

$$\mathrm{Arccot}(n^2+n+1) = \theta_n - \theta_{n+1}$$

Sum over $n = 1$ *to* M *and get*

$$\sum_{n=1}^{M} \mathrm{Arccot}(n^2+n+1) = \theta_1 - \theta_{M+1}$$

As $M \to \infty$ *we see* $\theta_M \to 0$ *so*

$$\sum_{n=0}^{\infty} \mathrm{Arccot}(n^2+n+1) = \mathrm{Arccot}(1) + \theta_1 = \pi/2.$$

A–4 A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

(a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.

(b) The sum of the n entries of a transversal is the same for all transversals of A.

An example of such a matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,$$

where the a_i's and b_i's are rational numbers.

Solution: *Consider a transversal sum including the terms* A_{ij} *and* $A_{i'j'}$. *Replacing these two terms in the sum by* $A_{ij'}$ *and* $A_{i'j}$ *gives another transversal with the same sum so*

$$A_{ij} + A_{i'j'} = A_{ij'} + A_{i'j}.$$

Rewrite this as

$$A_{ij} - A_{ij'} = A_{i'j} - A_{i'j'}$$

from which we see that the difference between column j *and column* j' *is the same in every row. Thus every column may be obtained from column 1 by adding the same number to each entry.*

If the entries in column 1 are all the same then we may satisfy the conditions of the problem by making every other column a constant. In each column there are 3 choices for the constant entry so there are 3^n *matrices of the desired form in which the first column is constant.*

There are $2^n - 2$ *possible first columns which are not constant and do not include the number -1. For each such we can make column* j *for each* $j \geq 2$ *by copying column 1 or by subtracting 1. Thus I can make 2 choices for each of* $n-1$ *columns generating*

$$2^{n-1}(2^n - 2)$$

matrices. There are the same number of matrices in which the first column is non constant and contains no 1s. This leaves

$$3^n - 2(2^n - 2) - 3 = 3^n - 2 \cdot 2^n + 1$$

ways to chooses column 1 so that the largest entry in column 1 is 1 and the smallest is -1. For these choices all other columns must be a copy of column 1. There are thus

$$f(n) = (3^n - 2 \cdot 2^n + 1) + 2 \cdot 2^{n-1} (2^n - 2) + 3^n = 2 \cdot 3^n + 4^n - 4 \cdot 2^n + 1$$

matrices with the desired property. Notice incidentally that if

$$A_{i1} = A_{ij} + c_j$$

for each i and j then the sum of any transversal is the sum of column 1 minus the sum of the c_j so that these processes build matrices of the desired form.

A–5 Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are functions of n real variables $x = (x_1, \ldots, x_n)$ with continuous second-order partial derivatives everywhere on $\mathbb{R}^n$. Suppose further that there are constants c_{ij} such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$

for all i and j, $1 \leq i \leq n$, $1 \leq j \leq n$. Prove that there is a function $g(x)$ on $\mathbb{R}^n$ such that $f_i + \partial g/\partial x_i$ is linear for all i, $1 \leq i \leq n$. (A linear function is one of the form

$$a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.)$$

Solution: *We will define*

$$h_i(x) = f_i - \sum_j b_{ij} x_j$$

for some constants b_{ij} chosen to make

$$\frac{\partial h_i}{\partial x_j} - \frac{\partial h_j}{\partial x_i} = 0.$$

This will guarantee that the function h with components h_i is the gradient of a potential g. The function g may be taken to be the line integral of h from the origin to the point x which is free of the path by virtue of the condition on h. It remains to find the constants b_{ij}. For a given set of constants we find

$$\frac{\partial h_i}{\partial x_j} - \frac{\partial h_j}{\partial x_i} = c_{ij} - (b_{ij} - b_{ji}).$$

Notice that

$$c_{ij} + c_{ji} = 0$$

and define $b_{ij} = c_{ij}/2$ to find

$$c_{ij} - (b_{ij} - b_{ji}) = c_{ij} - c_{ij}/2 + c_{ji}/2 = c_{ij} - c_{ij}/2 - c_{ij}/2 = 0.$$

This finishes the problem.

A–6 Let $a_1, a_2, \ldots, a_n$ be real numbers, and let $b_1, b_2, \ldots, b_n$ be distinct positive integers. Suppose that there is a polynomial $f(x)$ satisfying the identity

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1, b_2, \ldots, b_n$ and n (but independent of $a_1, a_2, \ldots, a_n$).

Solution: *For $n = 1$ we have*

$$f(x) = \frac{1 + ax^b}{1 - x}$$

which is a polynomial if and only if $1 - x$ divides the numerator which requires the numerator to vanish at $x = 1$. So $a = -1$ and

$$f(x) = 1 + x + \cdots + x^{b-1}$$

so that $f(1) = b$. In general the RHS has a 0 of order n at 1 so its derivatives up to order $n - 1$ vanish at $x = 1$. Moreover the value of f at 1 may be computed by applying l'Hôpital's rule n times. We get

$$\begin{aligned} -1 &= \sum a_i \\ 0 &= \sum a_i b_i \\ \vdots &= \vdots \\ 0 &= \sum a_i b_i (b_i - 1) \cdots (b_i - n + 1) \end{aligned}$$

We may write

$$b_i^l = b_i(b_i - 1) \cdots (b_i - l + 1) + b_i(b_i - 2) \cdots (b_i - l + 1) + 2b_i^2(b_i - 3) \cdots (b_i - l + 1) + \cdots$$

to see that

$$\begin{aligned} 1 &= \sum a_i \\ 0 &= \sum a_i b_i \\ \vdots &= \vdots \\ 0 &= \sum a_i b_i^{n-1} \end{aligned}$$

On the other hand if I differentiate the given formula n times and evaluate at $x = 1$ I get

$$n!(-1)^n f(1) = \sum a_i b_i (b_i - 1) \cdots (b_i - n)$$

The right hand side may be expanded as a sum of terms of the form $a_i b_i^l$ for $l \leq n$ So we get, for suitable constants c_l,

$$n!(-1)^n f(1) = \sum_{l=1}^{n} c_l \sum_i a_i b_i^l = \sum_i a_i b_i^n.$$

If B is the Vandermonde matrix of order n whose lth row contains entries b_i^{l-1} and A is the column vector of a_i then $BA = -e$ where e is the first natural basis vector in R^n. Since the b_i are distinct the matrix B in invertible and $A = -B^{-1}e$ determines the as from the bs. I now suspect the rest of the problem needs just an application of known theory of Vandermonde matrices!
Since the only non-zero entry in e is the first we need only compute the first column of B^{-1}. The jth entry in the first column of B^{-1} is the ratio of two determinants times $(-1)^{j+1}$. The denominator is the determinant Δ of the Vandermonde matrix B, namely,

$$\Delta = \prod_{i<j} (b_j - b_i).$$

The numerator is the determinant of the matrix obtained by striking out row 1 and column j from B. Each column of this matrix is of the form $b, b^2, \ldots, b^{n-1}$ with b being one of the b_i omitting $i = j$. Factoring out one power of b we find the numerator is

$$\prod_{i \neq j} b_i \Delta_j$$

where Δ_j is the determinant of the Vandermonde matrix of order $n-1$ with columns which are powers of b_k for $k \neq j$. To compute $\sum a_i b_i^n$ we multiply this element of the matrix inverse by -1 and by $(-1)^{j+1} b_j^n$ to get

$$\sum_i a_i b_i^n = \left[\sum b_i^{n-1}(-1)^{j+2}\frac{\Delta_j}{\Delta}\right]\prod b_i.$$

I claim that

$$\sum b_i^{n-1}(-1)^{j+1}\Delta_j = \Delta.$$

Now compute the determinant of B by expanding in minors about the elements of the last row. Striking out the last row and column j gives the Vandermonde matrix of order $n-1$ so that

$$\Delta = \sum_{j=1}^{n}(-1)^{n+j} b_j^{n-1}\Delta_j$$

It follows that

$$\sum_i a_i b_i^n = (-1)^n \prod b_j$$

and

$$f(1) = \prod b_j/n!$$

B–1 Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?

Solution: *We have $(b/2)^2 + (h/2)^2 = 1$. The area of the rectangle is bh. The height if the triangle is $1 - h/2$ and its base is b so the area of the triangle is $b(1-h/2)/2$ so*

$$h = (1-h/2)/2 \text{ or } 5h/2 = 1/2 \text{ or } h = 1/5$$

and

$$b = 2\sqrt{99/100} = 3\sqrt{11}/5.$$

Looks like I misunderstand!

B–2 Prove that there are only a finite number of possibilities for the ordered triple $T = (x-y, y-z, z-x)$, where x, y, z are complex numbers satisfying the simultaneous equations

$$x(x-1) + 2yz = y(y-1) + 2zx = z(z-1) + 2xy,$$

and list all such triples T.

Solution: *Let $u = y - x$ and $v = z - y$. We have*

$$x(x-1) + 2yz = x(x-1) + 2(u+x)(u+v+x) = x(x-1) + 2x^2 + (4u+2v-1)x + 2u(u+v),$$

$$y(y-1) + 2xz = (u+x)(u+x-1) + 2x(u+v+x) = x(x-1) + 2x^2 + (4u+2v-1)x + u(u-1)$$

and

$$z(z-1)+2xy = (u+v+x)(u+v+x-1)+2x(u+x) = x(x-1)+2x^2+(4u+2v-1)x+(u+v)(u+v-1).$$

Setting these three equal gives

$$2u(u+v) = u(u-1) = (u+v)(u+v-1).$$

If $u+v\neq 0$ *we get*

$$2u = u+v-1 \text{ or } v = 1+u.$$

This gives

$$u(u-1) = u^2 - u = (2u+1)(2u).$$

If $u\neq 0$ *then we see*

$$u-1 = 4u+2 \text{ or } 3u = -3$$

so $u=-1$, $v=1+u=0$ *is one solution. If* $u=0$ *then* $v=1$ *and it is easily checked that this is a solution. Finally there is the case* $u+v=0$ *or* $v=-u$ *giving* $u(u-1)=0$ *so* $u=-v=0$ *or* $u=1=-v$*. In summary the possible values of*

$$T = (x-y, y-z, z-x) = (-u, -v, u+v)$$

are (0,0,0), (-1,1,0), (1,0,-1), and (0,-1,1).

B–3 Let Γ consist of all polynomials in x with integer coefficienst. For f and g in Γ and m a positive integer, let $f\equiv g \pmod m$ mean that every coefficient of $f-g$ is an integral multiple of m. Let n and p be positive integers with p prime. Given that f, g, h, r and s are in Γ with $rf+sg\equiv 1 \pmod p$ and $fg\equiv h \pmod p$, prove that there exist F and G in Γ with $F\equiv f \pmod p$, $G\equiv g \pmod p$, and $FG\equiv h \pmod{p^n}$.

Solution: *We will construct* F *and* G *in the form*

$$F = f + \sum_1^{n-1} p^j q_j s$$

and

$$G = g + \sum_1^{n-1} p^j q_j r;$$

for some polynomials $q_1,\ldots,q_{n-1}$ *with integer coefficients. The hypotheses of the question guarantee that there are polynomials* t *and* u *with integer coefficients such that*

$$rf + sg = 1 + pt$$

and

$$fg = h + pu.$$

Then

$$FG = fg + \sum_1^{n-1} p^j q_j(rf+sg) + rs\sum_2^{2n-2} p^l v_l$$

where v_l *is the polynomial*

$$v_l = \sum_{j=1}^{l-1} q_j q_{j-j}.$$

Replace $rf+sg$ *and* fg *on the right hand side of* FG *to see*

$$FG = h + p(u+q_1) + \sum_2^{n-1} p^j(q_j + tq_{j-1} + rsv_j) + p^n w$$

where w is a polynomial with integer coefficients. Now define inductively

$$q_1 = -u$$

and

$$q_j = -tq_{j-1} - rsv_j$$

for $j = 2, \ldots, n-1$ to get

$$FG = h + p^n w$$

as desired. Notice that v_j is defined from $q_1, \ldots, q_{j-1}$ so that it may be used in the inductive definition of q_j.

B–4 For a positive real number r, let $G(r)$ be the minimum value of $|r - \sqrt{m^2 + 2n^2}|$ for all integers m and n. Prove or disprove the assertion that $\lim_{r\to\infty} G(r)$ exists and equals 0.

Solution: *The limit indicated is 0. For a given r let m be defined by $m \le r < m + 1$ and then define n by*

$$m^2 + 2n^2 \le r^2 < m^2 + 2(n+1)^2.$$

Since

$$\sqrt{m^2 + 2n^2} \le r < \sqrt{m^2 + 2(n+1)^2}$$

we see that

$$G(r) \le \sqrt{m^2 + 2(n+1)^2} - \sqrt{m^2 + 2n^2} = \sqrt{m^2 + 2n^2}\left(\sqrt{1 + (2n+1)/(m^2 + 2n^2)} - 1\right)$$

Note that

$$m \le \sqrt{m^2 + 2n^2} < m + 1$$

so that

$$2n^2 \le 2m + 1$$

and $2(n+1)^2 \le 4(n^2 + 1) < 4m + 6$ (using the inequality $(a+b)^2 \le 2(a^2 + b^2)$). It follows that

$$\frac{2n+1}{m^2 + 2n^2} \le \frac{\sqrt{4m+2}}{m^2 + 2n^2} \le \frac{\sqrt{6m}}{m^2} = \frac{\sqrt{6}}{m^{3/2}}.$$

Use a Taylor expansion of $\sqrt{1+x}$ to see that there is a constant c such that for all $x \le 6$ we have

$$\sqrt{1+x} - 1 \le \frac{c}{x}.$$

Then

$$G(r) \le (m+1)\frac{\sqrt{6}c}{m^{3/2}}$$

which clearly converges to 0.

B–5 Let $f(x, y, z) = x^2 + y^2 + z^2 + xyz$. Let $p(x, y, z), q(x, y, z), r(x, y, z)$ be polynomials with real coefficients satisfying

$$f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z).$$

Prove or disprove the assertion that the sequence p, q, r consists of some permutation of $\pm x, \pm y, \pm z$, where the number of minus signs is 0 or 2.

Suppose first that $p \equiv x$ and $q \equiv y$. Then r must satisfy

$$r^2 + xyr = z^2 + xyz.$$

Solving gives

$$r = \frac{-xy \pm \sqrt{x^2y^y + 4xyz + 4z^2}}{2} = \frac{-xy \pm \sqrt{(xy + 2z)^2}}{2}.$$

This gives two roots: $r \equiv z$ and

$$r = -z - xy.$$

Now for the latter case we find

$$f(p(x, y, z), q(x, y, z), r(x, y, z)) = x^2 + y^2 + (z + xy)^2 - xy(z + xy) = x^2 + y^2 + z^2 + xyz.$$

So the assertion is false, apparently, and totally surprisingly to me.

B–6 Suppose A, B, C, D are $n \times n$ matrices with entries in a field F, satisfying the conditions that AB^T and CD^T are symmetric and $AD^T - BC^T = I$. Here I is the $n \times n$ identity matrix, and if M is an $n \times n$ matrix, M^T is its transpose. Prove that $A^T D - C^T B = I$.

Solution: *Let M be the $2n \times 2n$ matrix with blocks A, B, C, D:*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and N be the $2n \times 2n$ matrix given by

$$N = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix}$$

Then

$$MN = \begin{bmatrix} AD^T - BC^T & BA^T - AB^T \\ CD^T - DC^T & DA^T - CB^T \end{bmatrix}$$

The symmetry conditions show that the two off-diagonal matrices vanish. We are given that the top left corner is I and the lower right hand corner is the transpose of this so it is also the identity. Thus MN is the $2n \times 2n$ identity matrix. Hence NM is the identity matrix since left inverses are right inverses. That is

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = NM = \begin{bmatrix} D^T A - C^T B & D^T B - B^T D \\ C^T A - A^T C & A^T D - C^T B \end{bmatrix}$$

This gives the desired conclusion in the lower right corner and the added information that $B^T D$ and $A^T C$ are both symmetric.