Brownian Motion

For fair random walk $Y_n = \text{number of heads minus number of tails},$

$$Y_n = U_1 + \cdots + U_n$$

where the $U_i$ are independent and

$$P(U_i = 1) = P(U_i = -1) = \frac{1}{2}$$

Notice:

$$\mathbb{E}(U_i) = 0$$
$$\text{Var}(U_i) = 1$$

Recall central limit theorem:

$$\frac{U_1 + \cdots + U_n}{\sqrt{n}} \Rightarrow N(0, 1)$$

Now: rescale time axis so that $n$ steps take 1 time unit and vertical axis so step size is $1/\sqrt{n}$. 
We now turn these pictures into a stochastic process:

For \( \frac{k}{n} \leq t < \frac{k+1}{n} \) we define

\[
X_n(t) = \frac{U_1 + \cdots + U_k}{\sqrt{n}}
\]

Notice:

\[
\mathbb{E}(X_n(t)) = 0
\]

and

\[
\text{Var}(X_n(t)) = \frac{k}{n}
\]

As \( n \to \infty \) with \( t \) fixed we see \( k/n \to t \). Moreover:

\[
\frac{U_1 + \cdots + U_k}{\sqrt{k}} = \sqrt{\frac{n}{k}}X_n(t)
\]

converges to \( N(0, 1) \) by the central limit theorem. Thus

\[
X_n(t) \Rightarrow N(0, t)
\]
Also: $X_n(t+s) - X_n(t)$ is independent of $X_n(t)$ because the 2 rvs involve sums of different $U_i$.

Conclusions.

As $n \to \infty$ the processes $X_n$ converge to a process $X$ with the properties:

1. $X(t)$ has a $N(0,t)$ distribution.

2. $X$ has independent increments: if
   \[
   0 = t_0 < t_1 < t_2 < \cdots < t_k
   \]
   then
   \[
   X(t_1) - X(t_0), \ldots, X(t_k) - X(t_{k-1})
   \]
   are independent.

3. The increments are **stationary**: for all $s$
   \[
   X(t + s) - X(s) \sim N(0,t)
   \]

4. $X(0) = 0$. 
**Def’n**: Any process satisfying 1-4 above is a Brownian motion.

**Properties of Brownian motion**

- Suppose \( t > s \). Then

\[
\begin{align*}
\mathbb{E}(X(t)|X(s)) &= \mathbb{E}\{X(t) - X(s) + X(s)|X(s)\} \\
&= \mathbb{E}\{X(t) - X(s)|X(s)\} \\
&\quad + \mathbb{E}\{X(s)|X(s)\} \\
&= 0 + X(s) = X(s)
\end{align*}
\]

Notice the use of independent increments and of \( \mathbb{E}(Y|Y) = Y \).

- Again if \( t > s \):

\[
\begin{align*}
\text{Var}\{X(t)|X(s)\} &= \text{Var}\{X(t) - X(s) + X(s)|X(s)\} \\
&= \text{Var}\{X(t) - X(s)|X(s)\} \\
&= \text{Var}\{X(t) - X(s)\} \\
&= t - s
\end{align*}
\]
Suppose $t < s$. Then $X(s) = X(t) + \{X(s) - X(t)\}$ is a sum of two independent normal variables. Do following calculation:

$X \sim N(0, \sigma^2)$, and $Y \sim N(0, \tau^2)$ independent. $Z = X + Y$.

Compute conditional distribution of $X$ given $Z$:

$$f_{X|Z}(x|z) = \frac{f_{X,Z}(x, z)}{f_Z(z)} = \frac{f_{X,Y}(x, z-x)}{f_Z(z)} = \frac{f_X(x)f_Y(z-x)}{f_Z(z)}$$

Now $Z$ is $N(0, \gamma^2)$ where $\gamma^2 = \sigma^2 + \tau^2$ so

$$f_{X|Z}(x|z) = \frac{1}{\sigma \sqrt{2\pi}} e^{-x^2/(2\sigma^2)} \frac{1}{\tau \sqrt{2\pi}} e^{-(z-x)^2/(2\tau^2)} \frac{1}{\gamma \sqrt{2\pi}} e^{-z^2/(2\gamma^2)}$$

$$= \frac{\gamma}{\tau \sigma \sqrt{2\pi}} \exp\{- (x - a)^2/(2b^2)\}$$

for suitable choices of $a$ and $b$. To find them compare coefficients of $x^2$, $x$ and 1.
Coefficient of $x^2$:

$$\frac{1}{b^2} = \frac{1}{\sigma^2} + \frac{1}{\tau^2}$$

so $b = \tau \sigma / \gamma$.

Coefficient of $x$:

$$\frac{a}{b^2} = \frac{z}{\tau^2}$$

so that

$$a = \frac{b^2 z}{\tau^2} = \frac{\sigma^2}{\sigma^2 + \tau^2 z}$$

Finally you should check that

$$\frac{a^2}{b^2} = \frac{z^2}{\tau^2} - \frac{z^2}{\gamma^2}$$

to make sure the coefficients of 1 work out as well.

Conclusion: given $Z = z$ the conditional distribution of $X$ is $N(a, b^2)$ with $a$ and $b$ as above.
Application to Brownian motion:

- For $t < s$ let $X$ be $X(t)$ and $Y$ be $X(s) - X(t)$ so $Z = X + Y = X(s)$. Then $\sigma^2 = t$, $\tau^2 = s - t$ and $\gamma^2 = s$. Thus

\[ b^2 = \frac{(s-t)t}{s} \]

and

\[ a = \frac{t}{s}X(s) \]

SO:

\[ \mathbb{E}(X(t)|X(s)) = \frac{t}{s}X(s) \]

and

\[ \text{Var}(X(t)|X(s)) = \frac{(s-t)t}{s} \]
The Reflection Principle

Tossing a fair coin:

HTHHHTHTHHTHHHTTHTH

5 more heads than tails

THTTTHTHTTHTTTTHHTHT

5 more tails than heads

Both sequences have the same probability.

So: for random walk starting at stopping time:

Any sequence with $k$ more heads than tails in next $m$ tosses is matched to sequence with $k$ more tails than heads. Both sequences have same prob.

Suppose $Y_n$ is a fair ($p = 1/2$) random walk. Define

$$M_n = \max\{Y_k, 0 \leq k \leq n\}$$
Compute $P(M_n \geq x)$? Trick: Compute

$$P(M_n \geq x, Y_n = y)$$

First: if $y \geq x$ then

$$\{M_n \geq x, Y_n = y\} = \{Y_n = y\}$$

Second: if $M_n \geq x$ then

$$T \equiv \min\{k : Y_k = x\} \leq n$$

Fix $y < x$. Consider a sequence of H’s and T’s which leads to say $T = k$ and $Y_n = y$.

Switch the results of tosses $k + 1$ to $n$ to get a sequence of H’s and T’s which has $T = k$ and $Y_n = x + (x - y) = 2x - y > x$. This proves

$$P(T = k, Y_n = y) = P(T = k, Y_n = 2x - y)$$
This is true for each \( k \) so

\[
P(M_n \geq x, Y_n = y) = P(M_n \geq x, Y_n = 2x - y) = P(Y_n = 2x - y)
\]

Finally, sum over all \( y \) to get

\[
P(M_n \geq x) = \sum_{y \geq x} P(Y_n = y) + \sum_{y < x} P(Y_n = 2x - y)
\]

Make the substitution \( k = 2x - y \) in the second sum to get

\[
P(M_n \geq x) = \sum_{y \geq x} P(Y_n = y) + \sum_{k > x} P(Y_n = k)
= 2 \sum_{k > x} P(Y_n = k) + P(Y_n = x)
\]
Brownian motion version:

\[ M_t = \max \{ X(s); 0 \leq s \leq t \} \]

\[ T_x = \min \{ s : X(s) = x \} \]

(called hitting time for level \( x \)). Then

\[ \{ T_x \leq t \} = \{ M_t \geq x \} \]

Any path with \( T_x = s < t \) and \( X(t) = y < x \) is matched to an equally likely path with \( T_x = s < t \) and \( X(t) = 2x - y > x \).

So for \( y > x \)

\[ P(M_t \geq x, X(t) > y) = P(X(t) > y) \]

while for \( y < x \)

\[ P(M_t \geq x, X(t) < y) = P(X(t) > 2x - y) \]
Let $y \to x$ to get

$$P(M_t \geq x, X(t) > x) = P(M_t \geq x, X(t) < x)$$
$$= P(X(t) > x)$$

Adding these together gives

$$P(M_t > x) = 2P(X(t) > x)$$
$$= 2P(N(0, 1) > x/\sqrt{t})$$

Hence $M_t$ has the distribution of $|N(0, t)|$. 
On the other hand in view of

\[\{T_x \leq t\} = \{M_t \geq x\}\]

the density of \(T_x\) is

\[\frac{d}{dt} 2P(N(0, 1) > x/\sqrt{t})\]

Use the chain rule to compute this. First

\[\frac{d}{dy} P(N(0, 1) > y) = -\phi(y)\]

where \(\phi\) is the standard normal density

\[\phi(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}}\]

because \(P(N(0, 1) > y)\) is 1 minus the standard normal cdf.
So
\[
\frac{d}{dt} 2P(N(0, 1) > x/\sqrt{t})
= -2\phi(x/\sqrt{t}) \frac{d}{dt}(x/\sqrt{t})
= \frac{x}{\sqrt{2\pi t^{3/2}}} \exp\{-x^2/(2t)\}
\]
This density is called the **Inverse Gaussian** density. \(T_x\) is called a **first passage time**

**NOTE:** the preceding is a density when viewed as a function of the variable \(t\).

**Martingales**

A stochastic process \(M(t)\) indexed by either a discrete or continuous time parameter \(t\) is a **martingale** if:
\[
\mathbb{E}\{M(t)|M(u); 0 \leq u \leq s\} = M(s)
\]
whenever \(s < t\).
Examples

- A fair random walk is a martingale.

- If $N(t)$ is a Poisson Process with rate $\lambda$ then $N(t) - \lambda t$ is a martingale.

- Standard Brownian motion (defined above) is a martingale.

Note: Brownian motion with drift is a process of the form

$$X(t) = \sigma B(t) + \mu t$$

where $B$ is standard Brownian motion, introduced earlier. $X$ is a martingale if $\mu = 0$. We call $\mu$ the drift.
• If $X(t)$ is a Brownian motion with drift then

$$Y(t) = e^{X(t)}$$

is a geometric Brownian motion. For suitable $\mu$ and $\sigma$ we can make $Y(t)$ a martingale.

• If a gambler makes a sequence of fair bets and $M_n$ is the amount of money s/he has after $n$ bets then $M_n$ is a martingale – even if the bets made depend on the outcomes of previous bets, that is, even if the gambler plays a strategy.
Some evidence for some of the above:

Random walk: $U_1, U_2, \ldots$ iid with

$$P(U_i = 1) = P(U_i = -1) = 1/2$$

and $Y_k = U_1 + \cdots + U_k$ with $Y_0 = 0$. Then

$$E(Y_n|Y_0, \ldots, Y_k)$$

$$= E(Y_n - Y_k + Y_k|Y_0, \ldots, Y_k)$$

$$= E(Y_n - Y_k|Y_0, \ldots, Y_k) + Y_k$$

$$= \sum_{k+1}^{n} E(U_j|U_1, \ldots, U_k) + Y_k$$

$$= \sum_{k+1}^{n} E(U_j) + Y_k$$

$$= Y_k$$
Things to notice:

$Y_k$ treated as constant given $Y_1, \ldots, Y_k$.

Knowing $Y_1, \ldots, Y_k$ is equivalent to knowing $U_1, \ldots, U_k$.

For $j > k$ we have $U_j$ independent of $U_1, \ldots, U_k$ so conditional expectation is unconditional expectation.

Since Standard Brownian Motion is limit of such random walks we get martingale property for standard Brownian motion.
**Poisson Process**: $X(t) = N(t) - \lambda t$. Fix $t > s$.

\[
\mathbb{E}(X(t)|X(u); 0 \leq u \leq s) = \mathbb{E}(N(t) - N(s) - \lambda(t - s)|\mathcal{H}_s) + X(s) = \mathbb{E}(N(t) - N(s)) - \lambda(t - s) + X(s) = \lambda(t - s) - \lambda(t - s) + X(s) = X(s)
\]

Things to notice:

I used independent increments.

$\mathcal{H}_s$ is shorthand for the conditioning event.

Similar to random walk calculation.
Black Scholes

We model the price of a stock as

\[ X(t) = x_0 e^{Y(t)} \]

where

\[ Y(t) = \sigma B(t) + \mu t \]

is a Brownian motion with drift (\( B \) is standard Brownian motion).

If annual interest rates are \( e^\alpha - 1 \) we call \( \alpha \) the instantaneous interest rate; if we invest $1 at time 0 then at time \( t \) we would have \( e^{\alpha t} \).

In this sense an amount of money \( x(t) \) to be paid at time \( t \) is worth only \( e^{-\alpha t} x(t) \) at time 0 (because that much money at time 0 will grow to \( x(t) \) by time \( t \)).
Present Value: If the stock price at time $t$ is $X(t)$ per share then the present value of 1 share to be delivered at time $t$ is

$$Z(t) = e^{-\alpha t} X(t)$$

With $X$ as above we see

$$Z(t) = x_0 e^{\sigma B(t) + (\mu - \alpha) t}$$

Now we compute

$$\mathbb{E} \{Z(t) | Z(u); 0 \leq u \leq s\}$$

$$= \mathbb{E} \{Z(t) | B(u); 0 \leq u \leq s\}$$

for $s < t$. Write

$$Z(t) = x_0 e^{\sigma B(s) + (\mu - \alpha) t} \times e^{\sigma (B(t) - B(s))}$$

Since $B$ has independent increments we find

$$\mathbb{E} \{Z(t) | B(u); 0 \leq u \leq s\}$$

$$= x_0 e^{\sigma B(s) + (\mu - \alpha) t} \times \mathbb{E} \left[ e^{\sigma (B(t) - B(s))} \right]$$

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Note: $B(t) - B(s)$ is $N(0, t - s)$; the expected value needed is the moment generating function of this variable at $\sigma$.

Suppose $U \sim N(0, 1)$. The Moment Generating Function of $U$ is

$$M_U(r) = \mathbb{E}(e^{rU}) = e^{r^2/2}$$

Rewrite

$$\sigma \{B(t) - B(s)\} = \sigma \sqrt{t - s}U$$

where $U \sim N(0, 1)$ to see

$$\mathbb{E} \left[ e^{\sigma \{B(t) - B(s)\}} \right] = e^{\sigma^2(t-s)/2}$$

Finally we get

$$\mathbb{E}\{Z(t)|Z(u); 0 \leq u \leq s\} = x_0 e^{\sigma B(s)} + (\mu - \alpha)s e^{(\mu - \alpha)(t-s) + \sigma^2(t-s)/2}$$

$$= Z(s)$$

provided

$$\mu + \sigma^2/2 = \alpha.$$
If this identity is satisfied then the present value of the stock price is a martingale.

**Option Pricing**

Suppose you can pay $c$ today for the right to pay $K$ for a share of this stock at time $t$ (regardless of the actual price at time $t$).

If, at time $t$, $X(t) > K$ you will **exercise** your **option** and buy the share making $X(t) - K$ dollars.

If $X(t) \leq K$ you will not exercise your option; it becomes worthless.

The present value of this option is

$$e^{-\alpha t}(X(t) - K)_+ - c$$

where

$$z_+ = \begin{cases} 
z & z > 0 \
0 & 0 \leq 0
\end{cases}$$

(Called **positive part** of $z$.)
In a fair market:

- The discounted share price $e^{-\alpha t}X(t)$ is a martingale.

- The expected present value of the option is 0.

So:

$$c = e^{-\alpha t}E\left[\{X(t) - K\}_+\right]$$

Since

$$X(t) = x_0e^{N(\mu t, \sigma^2 t)}$$

we are to compute

$$E\left\{\left(x_0e^{\sigma t^{1/2}U} + \mu t - K\right)_+\right\}$$
This is
\[ \int_a^\infty (x_0 e^{bu+d} - K) \, e^{-u^2/2} \, du/\sqrt{2\pi} \]
where
\[ a = (\log(K/x_0) - \mu t)/(\sigma t^{1/2}) \]
\[ b = \sigma t^{1/2} \]
\[ d = \mu t \]

Evidently
\[ K \int_a^\infty e^{-u^2/2} \, du/\sqrt{2\pi} = KP(N(0, 1) > a) \]

The other integral needed is
\[ \int_a^\infty e^{-u^2/2 + bu} \, du/\sqrt{2\pi} \]
\[ = \int_a^\infty e^{-(u-b)^2/2} e^{b^2/2} \sqrt{2\pi} du \]
\[ = \int_{a-b}^\infty e^{-v^2/2} e^{b^2/2} \sqrt{2\pi} dv \]
\[ = e^{b^2/2} P(N(0, 1) > a-b) \]
Introduce the notation

\[ \Phi(v) = P(N(0, 1) \leq v) = P(N(0, 1) > -v) \]

and do all the algebra to get

\[
c = \left\{ e^{-\alpha t} e^{b^2/2 + d} x_0 \Phi(b - a) - Ke^{-\alpha t} \Phi(-a) \right\}
\]

\[
= x_0 e^{(\mu + \sigma^2/2 - \alpha) t} \Phi(b - a) - Ke^{-\alpha t} \Phi(-a)
\]

\[
= x_0 \Phi(b - a) - Ke^{-\alpha t} \Phi(-a)
\]

This is the Black-Scholes option pricing formula.