The ideas of the previous sections can be used to prove the basic sampling theory results for the normal family. Here is the theorem which describes the distribution theory of the most important statistics.

Theorem 1 *Suppose $X_1, \ldots, X_n$ are independent $N(\mu, \sigma^2)$ random variables. Then*

1. *$\bar{X}$ (sample mean)and s^2 (sample variance) independent.*
2. *$n^{1/2}(\bar{X} - \mu)/\sigma \sim N(0, 1)$.*
3. *$(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$.*
4. *$n^{1/2}(\bar{X} - \mu)/s \sim t_{n-1}$.*

Proof: Let $Z_i = (X_i - \mu)/\sigma$. Then $Z_1, \ldots, Z_p$ are independent $N(0, 1)$. So $Z = (Z_1, \ldots, Z_p)^t$ is multivariate standard normal.

Note that $\bar{X} = \sigma\bar{Z} + \mu$ and $s^2 = \sum(X_i - \bar{X})^2/(n-1) = \sigma^2 \sum(Z_i - \bar{Z})^2/(n-1)$ Thus

$$\frac{n^{1/2}(\bar{X} - \mu)}{\sigma} = n^{1/2}\bar{Z}$$

$$\frac{(n-1)s^2}{\sigma^2} = \sum(Z_i - \bar{Z})^2$$

and

$$T = \frac{n^{1/2}(\bar{X} - \mu)}{s} = \frac{n^{1/2}\bar{Z}}{s_Z}$$

where $(n-1)s_Z^2 = \sum(Z_i - \bar{Z})^2$. It is therefore enough to prove the theorem in the case $\mu = 0$ and $\sigma = 1$.

Step 1: Define

$$Y = (\sqrt{n}\bar{Z}, Z_1 - \bar{Z}, \ldots, Z_{n-1} - \bar{Z})^t\,.$$

(So that Y has same dimension as Z.) Now

$$Y = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ 1 - \frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1 - \frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix}$$

or letting M denote the matrix

$$Y = MZ\,.$$

It follows that $Y \sim MVN(0, MM^t)$ so we need to compute MM^t:

$$\begin{aligned}
MM^t &= \left[\begin{array}{c|cccc}
1 & 0 & 0 & \cdots & 0 \\ \hline
0 & 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\
\vdots & -\frac{1}{n} & \ddots & \cdots & -\frac{1}{n} \\
0 & \vdots & \cdots & & 1-\frac{1}{n}
\end{array}\right] \\
&= \left[\begin{array}{c|c}
1 & 0 \\ \hline
& \\
0 & Q
\end{array}\right].
\end{aligned}$$

Solve for Z from Y: $Z_i = n^{-1/2}Y_1 + Y_{i+1}$ for $1 \le i \le n-1$. Use the identity

$$\sum_{i=1}^{n} (Z_i - \bar{Z}) = 0$$

to get $Z_n = -\sum_{i=2}^{n} Y_i + n^{-1/2}Y_1$. So M is invertible:

$$\Sigma^{-1} \equiv (MM^t)^{-1} = \left[\begin{array}{c|c}
1 & 0 \\ \hline
& \\
0 & Q^{-1}
\end{array}\right].$$

Now use the change of variables formula to find f_Y. Let $\mathbf{y}_2$ denote the vector whose entries are $y_2, \ldots, y_n$. Note that

$$y^t \Sigma^{-1} y = y_1^2 + \mathbf{y}_2^t Q^{-1} \mathbf{y}_2\,.$$

Then

$$\begin{aligned}
f_Y(y) =& (2\pi)^{-n/2} \exp[-y^t\Sigma^{-1}y/2]/|\det M| \\
=& \frac{1}{\sqrt{2\pi}} e^{-y_1^2/2} \times \\
& \frac{(2\pi)^{-(n-1)/2} \exp[-\mathbf{y}_2^t Q^{-1} \mathbf{y}_2/2]}{|\det M|}\,.
\end{aligned}$$

Note: f_Y is a function of y_1 times a ftn of $y_2, \ldots, y_n$. Thus $\sqrt{n}\bar{Z}$ is independent of $Z_1 - \bar{Z}, \ldots, Z_{n-1} - \bar{Z}$. Since s_Z^2 is a function of $Z_1 - \bar{Z}, \ldots, Z_{n-1} - \bar{Z}$ we see that $\sqrt{n}\bar{Z}$ and s_Z^2 are independent.

Also, the density of Y_1 is a multiple of the function of y_1 in the factorization above. But this factor is a standard normal density so $\sqrt{n}\bar{Z} \sim N(0,1)$.

The first 2 parts of the theorem are now done. The third part is a homework exercise.

I now present a derivation of the χ^2 density; this is not part of the proof of the theorem but is another distribution theory example. Suppose $Z_1, \ldots, Z_n$ are independent $N(0,1)$. Define the χ_n^2 distribution to be that of $U = Z_1^2 + \cdots + Z_n^2$. Define angles $\theta_1, \ldots, \theta_{n-1}$ by

$$
\begin{aligned}
Z_1 &= U^{1/2} \cos\theta_1 \\
Z_2 &= U^{1/2} \sin\theta_1 \cos\theta_2 \\
\vdots &= \vdots \\
Z_{n-1} &= U^{1/2} \sin\theta_1 \cdots \sin\theta_{n-2} \cos\theta_{n-1} \\
Z_n &= U^{1/2} \sin\theta_1 \cdots \sin\theta_{n-1} \,.
\end{aligned}
$$

(These are k spherical co-ordinates in n dimensions. The θ values run from 0 to π except last θ from 0 to 2π.) Here are the derivative formulae:

$$
\frac{\partial Z_i}{\partial U} = \frac{1}{2U} Z_i
$$

and

$$
\frac{\partial Z_i}{\partial \theta_j} = \begin{cases} 0 & j > i \\ -Z_i \tan\theta_i & j = i \\ Z_i \cot\theta_j & j < i \,. \end{cases}
$$

Fix $n = 3$ to clarify the formulae. Use the shorthand $R = \sqrt{U}$ The matrix of partial derivatives is

$$
\begin{bmatrix}
\frac{\cos\theta_1}{2R} & -R\sin\theta_1 & 0 \\
\frac{\sin\theta_1\cos\theta_2}{2R} & R\cos\theta_1\cos\theta_2 & -R\sin\theta_1\sin\theta_2 \\
\frac{\sin\theta_1\sin\theta_2}{2R} & R\cos\theta_1\sin\theta_2 & R\sin\theta_1\cos\theta_2
\end{bmatrix} .
$$

We can find the determinant by adding $2U^{1/2}\cos\theta_j/\sin\theta_j$ times col 1 to col $j+1$ (no change in the determinant). The resulting matrix is lower triangular with diagonal entries given by

$$\frac{\cos\theta_1}{R}, \frac{R\cos\theta_2}{\cos\theta_1}, \frac{R\sin\theta_1}{\cos\theta_2}$$

Multiply these together to get

$$U^{1/2}\sin(\theta_1)/2$$

which I observe is non-negative for all U and θ_1. For general n every term in the first column contains a factor $U^{-1/2}/2$ while every other entry has a factor $U^{1/2}$.

Fact: multiplying a column in a matrix by c multiplies the determinant by c.

So: the Jacobian of the transformation is

$$u^{(n-1)/2}u^{-1/2}/2 \times h(\theta_1, \theta_{n-1})$$

for some function, h, which depends only on the angles. Thus the joint density of $U, \theta_1, \ldots \theta_{n-1}$ is

$$(2\pi)^{-n/2}\exp(-u/2)u^{(n-2)/2}h(\theta_1, \cdots, \theta_{n-1})/2\,.$$

To compute the density of U we must do an $n-1$ dimensional multiple integral $d\theta_{n-1}\cdots d\theta_1$.

The answer has the form

$$cu^{(n-2)/2}\exp(-u/2)$$

for some c. We can evaluate c by making

$$\begin{aligned}\int f_U(u)du &= c\int_0^\infty u^{(n-2)/2}\exp(-u/2)du \\ &= 1.\end{aligned}$$

Substitute $y = u/2$, $du = 2dy$ to see that

$$\begin{aligned}c2^{n/2}\int_0^\infty y^{(n-2)/2}e^{-y}dy &= c2^{n/2}\Gamma(n/2) \\ &= 1.\end{aligned}$$

Conclusion: the χ_n^2 density is

$$\frac{1}{2\Gamma(n/2)}\left(\frac{u}{2}\right)^{(n-2)/2} e^{-u/2}1(u>0)\,.$$

The fourth part of the theorem is a consequence of first 3 parts and the definition of the t_ν distribution.

Definition: $T \sim t_\nu$ if T has same distribution as

$$Z/\sqrt{U/\nu}$$

for $Z \sim N(0,1)$, $U \sim \chi_\nu^2$ and Z, U independent.

Though the proof of the theorem is now finished I will Derive the density of T in this definition as a further example of the techniques of distribution theory. Begin with the cumulative distribution function of T written in terms of Z and U:

$$\begin{aligned} P(T \le t) &= P(Z \le t\sqrt{U/\nu}) \\ &= \int_0^\infty \int_{-\infty}^{t\sqrt{u/\nu}} f_Z(z) f_U(u) dz du \end{aligned}$$

Differentiate this cdf with respect to t by differentiating the inner integral:

$$\frac{\partial}{\partial t}\int_{at}^{bt} f(x)dx = bf(bt) - af(at)$$

by the fundamental theorem of calculus. Hence

$$\frac{d}{dt}P(T \le t) = \int_0^\infty \frac{f_U(u)}{\sqrt{2\pi}}\left(\frac{u}{\nu}\right)^{1/2}\exp\left(-\frac{t^2 u}{2\nu}\right) du\,.$$

Plug in

$$f_U(u) = \frac{1}{2\Gamma(\nu/2)}(u/2)^{(\nu-2)/2}e^{-u/2}$$

to get

$$f_T(t) = \frac{\int_0^\infty (u/2)^{(\nu-1)/2} e^{-u(1+t^2/\nu)/2} du}{2\sqrt{\pi\nu}\Gamma(\nu/2)}\,.$$

Substitute $y = u(1+t^2/\nu)/2$, to get

$$dy = (1+t^2/\nu)du/2$$

$$(u/2)^{(\nu-1)/2} = [y/(1+t^2/\nu)]^{(\nu-1)/2}$$

leading to

$$f_T(t) = \frac{(1+t^2/\nu)^{-(\nu+1)/2}}{\sqrt{\pi\nu}\,\Gamma(\nu/2)} \int_0^\infty y^{(\nu-1)/2} e^{-y} dy$$

or

$$f_T(t) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\,\Gamma(\nu/2)} \frac{1}{(1+t^2/\nu)^{(\nu+1)/2}}\,.$$