# STAT 830
# Likelihood Methods

Richard Lockhart

SFU

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# Purposes of These Notes

- Define the likelihood, log-likelihood and score functions.
- Summarize likelihood methods
- Describe maximum likelihood estimation
- Give sequence of examples.

# Likelihood Methods of Inference

- Toss a thumb tack 6 times and imagine it lands point up twice.
- Let $p$ be probability of landing points up.
- Probability of getting exactly 2 point up is

$$15p^2(1-p)^4$$

- This function of $p$, is the **likelihood** function.
- **Definition**: The likelihood function is map $L$: domain $\Theta$, values given by

$$L(\theta) = f_\theta(X)$$

- Key Point: think about how the density depends on $\theta$ not about how it depends on $X$.
- Notice: $X$, observed value of the data, has been plugged into the formula for density.
- Notice: coin tossing example uses the discrete density for $f$.
- We use likelihood for most inference problems:

# List of likelihood techniques

- Point estimation: we must compute an estimate $\hat\theta = \hat\theta(X)$ which lies in $\Theta$. The **maximum likelihood estimate (MLE)** of $\theta$ is the value $\hat\theta$ which maximizes $L(\theta)$ over $\theta \in \Theta$ if such a $\hat\theta$ exists.
- Point estimation of a function of $\theta$: we must compute an estimate $\hat\phi = \hat\phi(X)$ of $\phi = g(\theta)$. We use $\hat\phi = g(\hat\theta)$ where $\hat\theta$ is the MLE of $\theta$.
- Interval (or set) estimation. We must compute a set $C = C(X)$ in $\Theta$ which we think will contain $\theta_0$. We will use

$$\{\theta \in \Theta : L(\theta) > c\}$$

  for a suitable $c$.
- Hypothesis testing: decide whether or not $\theta_0 \in \Theta_0$ where $\Theta_0 \subset \Theta$. We base our decision on the likelihood ratio

$$\frac{\sup\{L(\theta); \theta \in \Theta \setminus \Theta_0\}}{\sup\{L(\theta); \theta \in \Theta_0\}}.$$

# Maximum Likelihood Estimation

- To find MLE maximize $L$.
- Typical function maximization problem:
- Set gradient of $L$ equal to 0.
- Check root is maximum, not minimum or saddle point.
- Examine some likelihood plots in examples:
- Focus on fact that each data set corresponds to its own function of $\theta$
- So the graph itself is a *statistic*.

# Cauchy Data

- IID sample $X_1, \ldots, X_n$ from Cauchy($\theta$) density

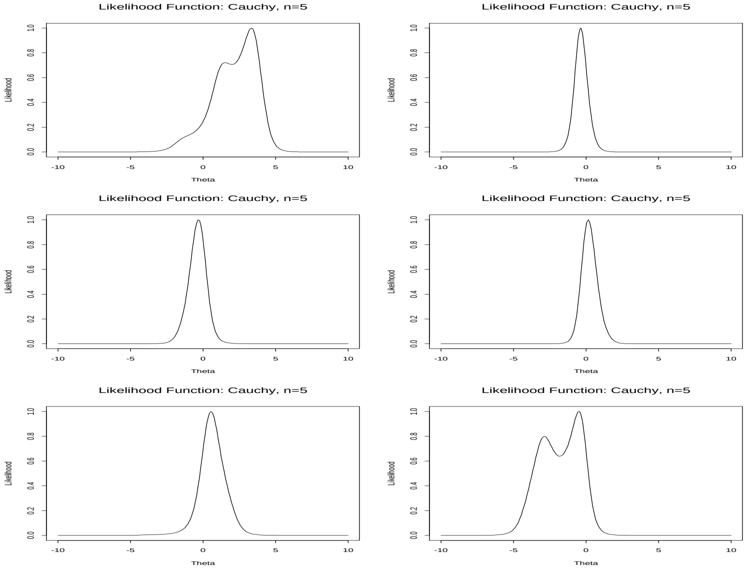
$$f(x;\theta) = \frac{1}{\pi(1+(x-\theta)^2)}$$

- The likelihood function is

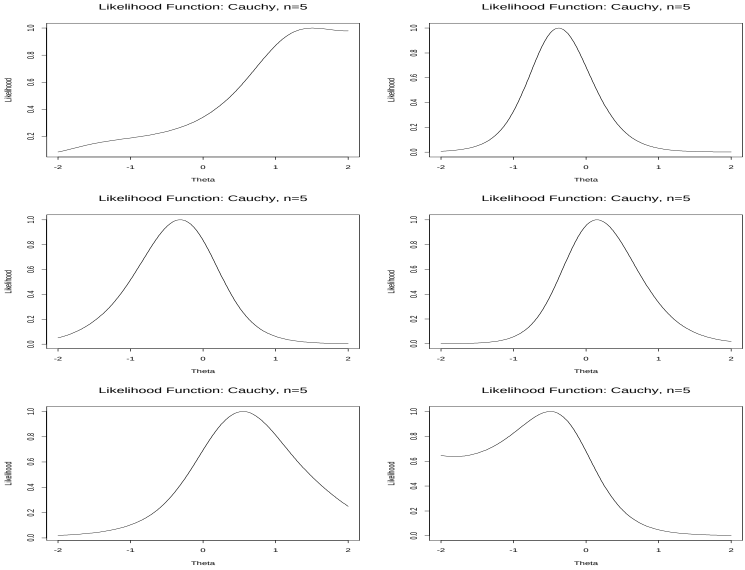
$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\pi(1+(X_i-\theta)^2)}$$

- Here are some likelihood plots.

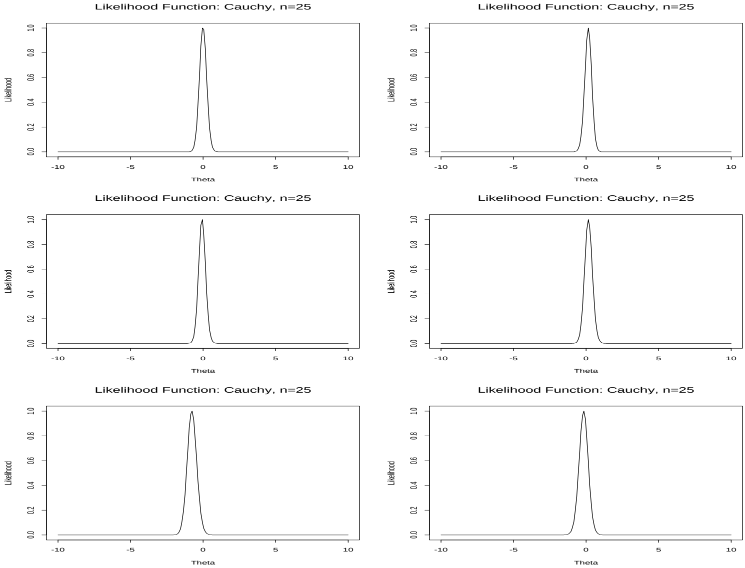
# Cauchy data $n = 5$



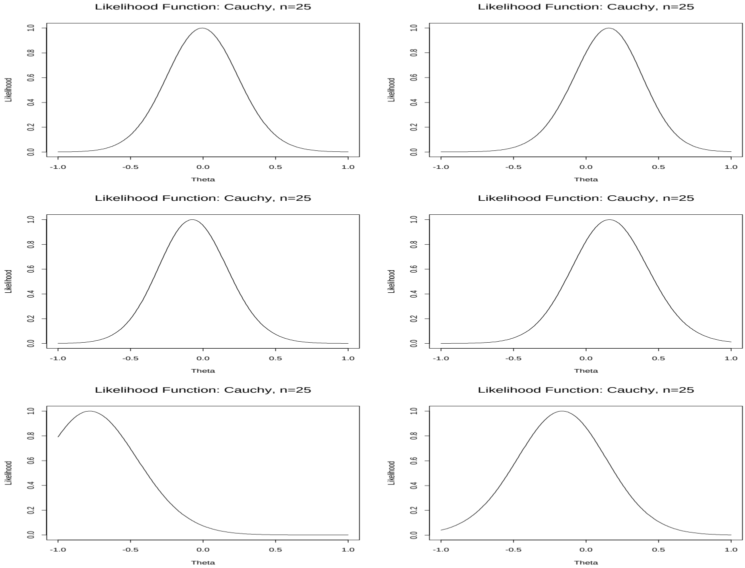
# Cauchy data $n = 5$ — close-up



# Cauchy data $n = 25$ up



# Cauchy data $n = 25$ — close-up



# Things to see in the plots

- The likelihood functions have peaks near the true value of $\theta$ (which is 0 for the data sets I generated).
- The peaks are narrower for the larger sample size.
- The peaks have a more regular shape for the larger value of $n$.
- I actually plotted $L(\theta)/L(\hat{\theta})$ which has exactly the same shape as $L$ but runs from 0 to 1 on the vertical scale.

# The log-likelihood

- To maximize this likelihood: differentiate $L$, set result equal to 0.
- Notice $L$ is product of $n$ terms; derivative is

$$\sum_{i=1}^{n} \prod_{j \neq i} \frac{1}{\pi(1+(X_j-\theta)^2)} \frac{2(X_i-\theta)}{\pi(1+(X_i-\theta)^2)^2}$$

  which is quite unpleasant.
- Much easier to work with logarithm of $L$: log of product is sum and logarithm is monotone increasing.
- **Definition**: The **Log Likelihood** function is

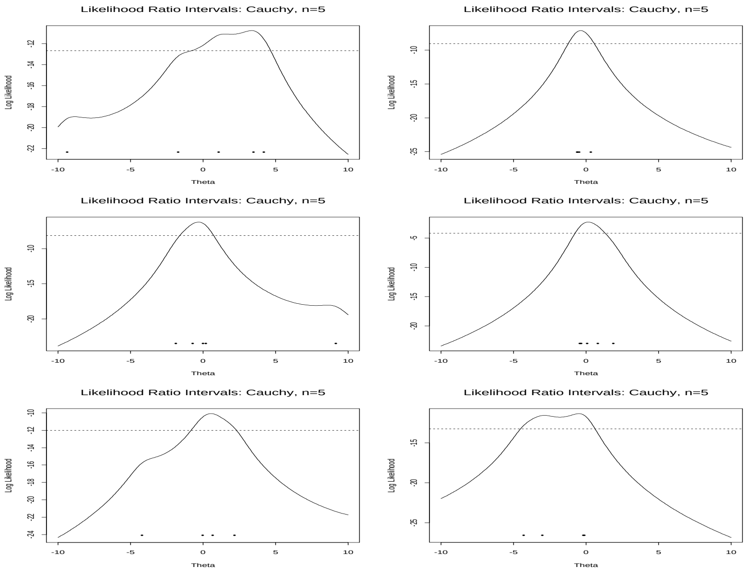
$$\ell(\theta) = \log\{L(\theta)\}.$$

- For the Cauchy problem we have

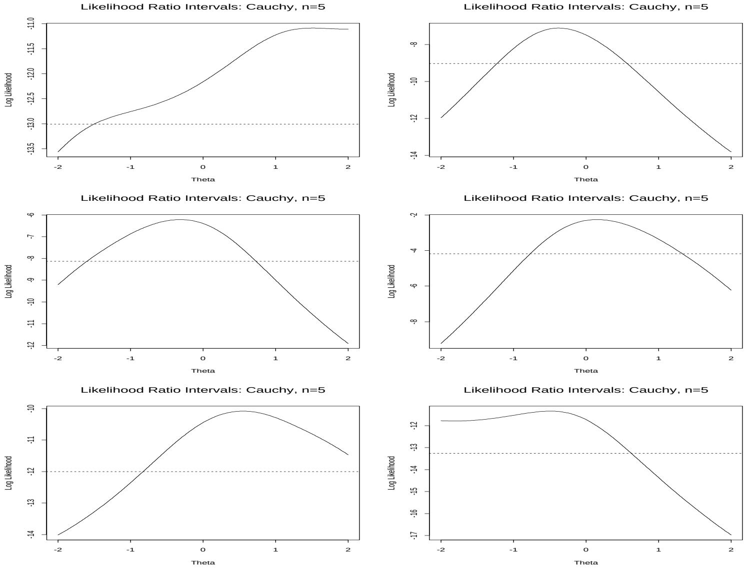
$$\ell(\theta) = -\sum \log(1+(X_i-\theta)^2) - n\log(\pi)$$

- Now we examine log likelihood plots.

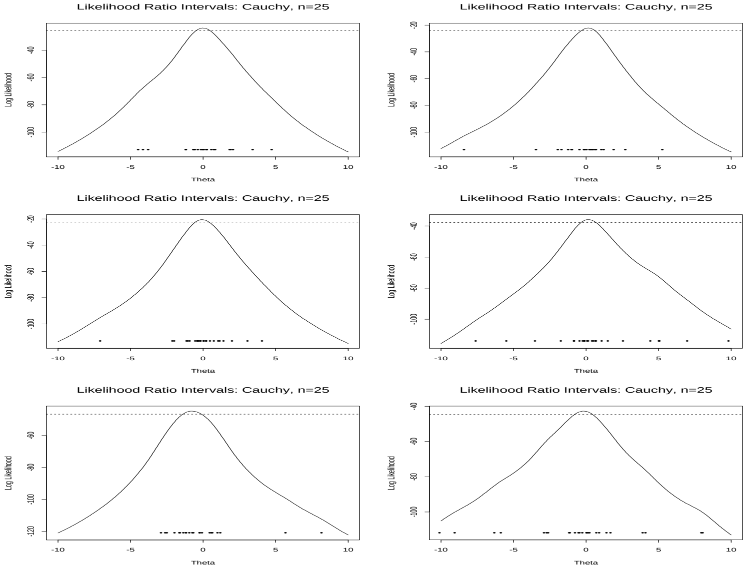
# Cauchy log-likelihood $n = 5$



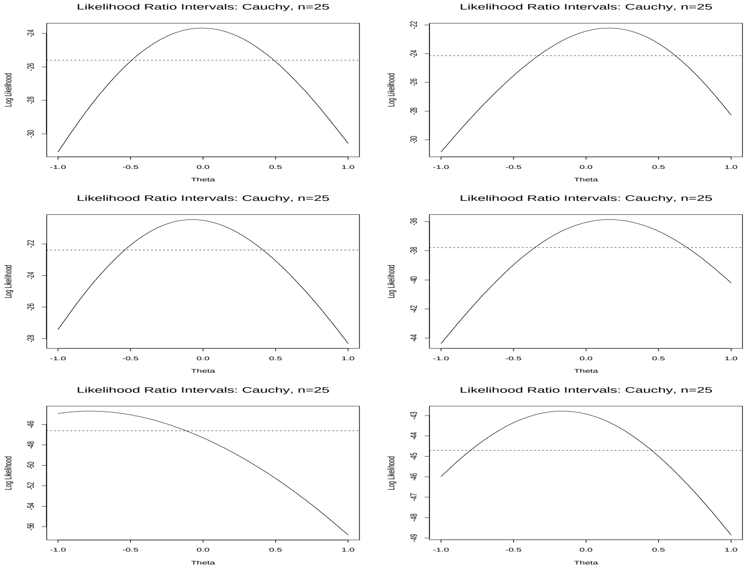
# Cauchy log-likelihood $n = 5$, close-up



# Cauchy log-likelihood $n = 25$



# Cauchy log-likelihood $n = 25$, close-up



# Things to notice

- Plots of $\ell$ for $n = 25$ quite smooth, rather parabolic.
- For $n = 5$ many local maxima and minima of $\ell$.
- Likelihood tends to 0 as $|\theta| \to \infty$ so max of $\ell$ occurs at a root of $\ell'$, derivative of $\ell$ wrt $\theta$.
- **Definition**: **Score Function** is gradient of $\ell$

$$U(\theta) = \frac{\partial \ell}{\partial \theta}$$

- MLE $\hat{\theta}$ usually root of **Likelihood Equations**

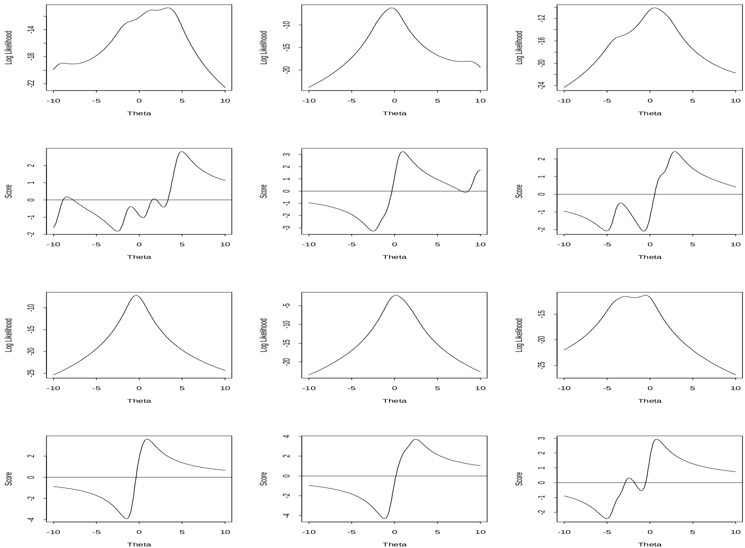
$$U(\theta) = 0$$

- In our Cauchy example we find

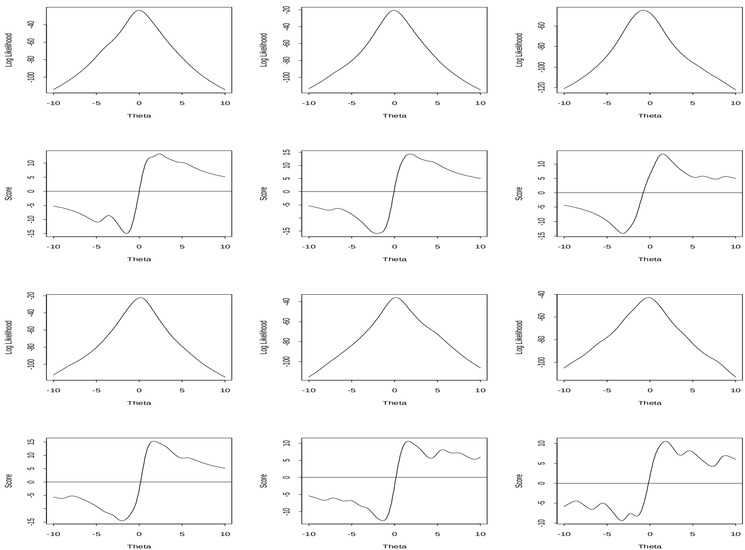
$$U(\theta) = \sum \frac{2(X_i - \theta)}{1 + (X_i - \theta)^2}$$

- Now we examine plots of score functions.
- Notice: often multiple roots of likelihood equations.

# Cauchy score $n = 5$



# Cauchy score $n = 25$



# Binomial example

- **Example**: $X \sim \text{Binomial}(n, \theta)$

$$L(\theta) = \binom{n}{X} \theta^X (1-\theta)^{n-X}$$

$$\ell(\theta) = \log\binom{n}{X} + X\log(\theta) + (n-X)\log(1-\theta)$$

$$U(\theta) = \frac{X}{\theta} - \frac{n-X}{1-\theta}$$

- The function $L$ is 0 at $\theta = 0$ and at $\theta = 1$ unless $X = 0$ or $X = n$ so for $1 \le X < n$ the MLE must be found by setting $U = 0$ and getting

$$\hat{\theta} = \frac{X}{n}$$

# Binomial Continued

- For $X = n$ the log-likelihood has derivative

$$U(\theta) = \frac{n}{\theta} > 0$$

  for all $\theta$

- So the likelihood is an increasing function of $\theta$ which is maximized at $\hat{\theta} = 1 = X/n$.
- Similarly when $X = 0$ the maximum is at $\hat{\theta} = 0 = X/n$.
- In all cases

$$\hat{\theta} = \frac{X}{n}.$$

# The Normal Distribution

- Now we have $X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$.
- There are two parameters $\theta = (\mu, \sigma)$.
- We find

$$L(\mu,\sigma) = \frac{e^{-\sum(X_i-\mu)^2/(2\sigma^2)}}{(2\pi)^{n/2}\sigma^n}$$

$$\ell(\mu,\sigma) = -\frac{n}{2}\log(2\pi) - \frac{\sum(X_i-\mu)^2}{2\sigma^2} - n\log(\sigma)$$

  and that $U$ is

$$\left[\begin{array}{c} \frac{\sum(X_i-\mu)}{\sigma^2} \\ \frac{\sum(X_i-\mu)^2}{\sigma^3} - \frac{n}{\sigma} \end{array}\right]$$

- Notice that $U$ is a function with two components because $\theta$ has two components.
- Setting the likelihood equal to 0 and solving gives

$$\hat{\mu} = \bar{X} \quad \text{and} \quad \hat{\sigma} = \sqrt{\frac{\sum(X_i-\bar{X})^2}{n}}.$$

# Normal example continued

- Check this is maximum by computing one more derivative.
- Matrix $H$ of second derivatives of $\ell$ is

$$\begin{bmatrix} \frac{-n}{\sigma^2} & \frac{-2\sum(X_i-\mu)}{\sigma^3} \\ \frac{-2\sum(X_i-\mu)}{\sigma^3} & \frac{-3\sum(X_i-\mu)^2}{\sigma^4}+\frac{n}{\sigma^2} \end{bmatrix}$$

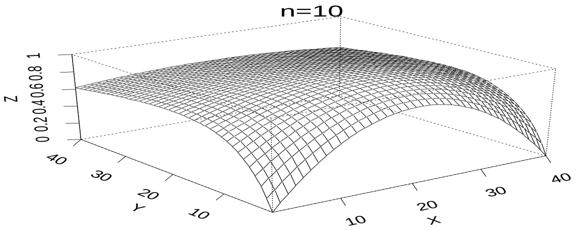
- Plugging in the mle gives

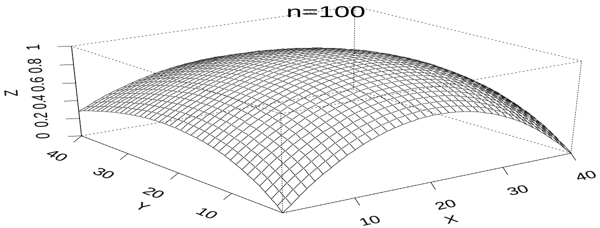
$$H(\hat\theta) = \begin{bmatrix} \frac{-n}{\hat\sigma^2} & 0 \\ 0 & \frac{-2n}{\hat\sigma^2} \end{bmatrix}$$

which is negative definite.

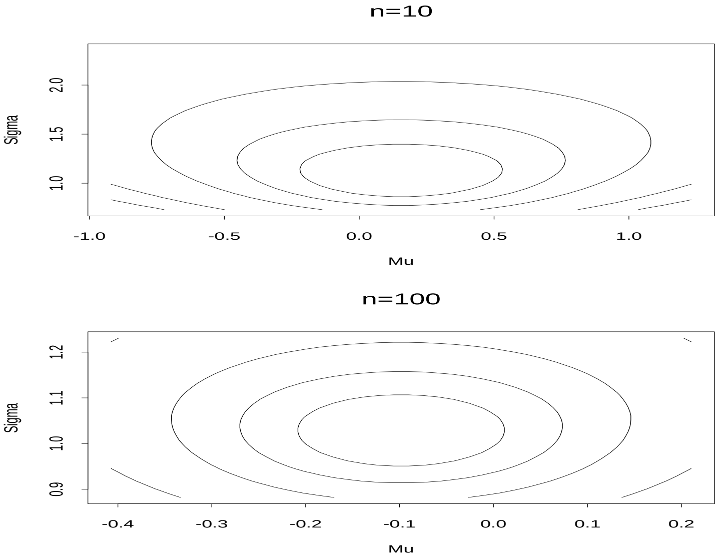
- Both its eigenvalues are negative.
- So $\hat\theta$ must be a local maximum.
- Examine contour and perspective plots of $\ell$.

# Normal likelihood perspective plot



# Normal likelihood perspective plot



# Observations

- Notice that the contours are quite ellipsoidal for the larger sample size.
- For $X_1, \ldots, X_n$ iid log likelihood is

$$\ell(\theta) = \sum \log(f(X_i, \theta)) .$$

- The score function is

$$U(\theta) = \sum \frac{\partial \log f}{\partial \theta}(X_i, \theta) .$$

  MLE $\hat{\theta}$ maximizes $\ell$.

- If maximum occurs in interior of parameter space and the log likelihood continuously differentiable then $\hat{\theta}$ solves the likelihood equations

$$U(\theta) = 0 .$$

# Solving $U(\theta) = 0$: Examples

- **N**$(\mu, \sigma^2)$
- Unique root of likelihood equations is a global maximum.
- **Remark**: Suppose we called $\tau = \sigma^2$ the parameter.
  - Score function still has two components: first component same as before but second component is

$$\frac{\partial}{\partial \tau} \ell = \frac{\sum (X_i - \mu)^2}{2\tau^2} - \frac{n}{2\tau}$$

  - Setting the new likelihood equations equal to 0 still gives

$$\hat{\tau} = \hat{\sigma}^2$$

  - General **invariance** (or **equivariance**) principal:
  - If $\phi = g(\theta)$ is some reparametrization of a model (a one to one relabelling of the parameter values) then $\hat{\phi} = g(\hat{\theta})$.
  - Does not apply to other estimators.

# Examples

- **Cauchy, location** $\theta$
- At least 1 root of likelihood equations but often several more.
- One root is a global maximum; others, if they exist may be local minima or maxima.
- **Binomial(**$n, \theta$**)**
- If $X = 0$ or $X = n$: no root of likelihood equations; likelihood is monotone.
- Other values of $X$: unique root, a global maximum.
- Global maximum at $\hat{\theta} = X/n$ even if $X = 0$ or $n$.

# Examples: 2 parameter exponential

- The density is

$$f(x;\alpha,\beta) = \frac{1}{\beta} e^{-(x-\alpha)/\beta} 1(x > \alpha)$$

- Log-likelihood is $-\infty$ for $\alpha > \min\{X_1,\ldots,X_n\}$ and otherwise is

$$\ell(\alpha,\beta) = -n\log(\beta) - \sum (X_i - \alpha)/\beta$$

- Increasing function of $\alpha$ till $\alpha$ reaches

$$\hat\alpha = X_{(1)} = \min\{X_1,\ldots,X_n\}$$

  which gives mle of $\alpha$.

- Now plug in $\hat\alpha$ for $\alpha$; get *profile likelihood* for $\beta$:

$$\ell_{\text{profile}}(\beta) = -n\log(\beta) - \sum (X_i - X_{(1)})/\beta$$

# 2 parameter exponential continued

- Set $\beta$ derivative equal to 0 to get

$$\hat{\beta} = \sum (X_i - X_{(1)})/n$$

- Notice mle $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$ does *not* solve likelihood equations; we had to look at the edge of the possible parameter space.
- $\alpha$ is called a *support* or *truncation* parameter.
- ML methods behave oddly in problems with such parameters.

# Three parameter Weibull

- The density in question is

$$f(x;\alpha,\beta,\gamma) = \frac{1}{\beta}\left(\frac{x-\alpha}{\beta}\right)^{\gamma-1} \times \exp[-\{(x-\alpha)/\beta\}^{\gamma}]1(x>\alpha)$$

- Three likelihood equations:
- Set $\beta$ derivative equal to 0; get

$$\hat{\beta}(\alpha,\gamma) = \left[\sum (X_i-\alpha)^{\gamma}/n\right]^{1/\gamma}$$

  where $\hat{\beta}(\alpha,\gamma)$ indicates mle of $\beta$ could be found by finding the mles of the other two parameters and then plugging in to the formula above.
- No explicit solution for remaining par ests; numerical methods needed.
- But putting $\gamma < 1$ and letting $\alpha \to X_{(1)}$ will make the log likelihood go to $\infty$.
- MLE is not uniquely defined: any $\gamma < 1$ and any $\beta$ will do.

# Three parameter Weibull continued

- Subscript 0 indicates true parameter values.
- If $\gamma_0 > 1$ then probability that there is a root of the likelihood equations is high.
- In this case there must be two more roots: a local maximum and a saddle point!
- For $\gamma_0 > 1$ theory to come applies to the local maximum and not to the global maximum of the likelihood equations.