Inversion

Previous theorem is non-constructive characterization.

Can get from ϕ_X to F_X or f_X by **inversion**.

See homework for basic **inversion** formula:

If X is a random variable taking only integer values then for each integer k

$$\begin{aligned} P(X=k) &= \frac{1}{2\pi}\int_0^{2\pi} \phi_X(t)e^{-itk}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_X(t)e^{-itk}dt\,. \end{aligned}$$

The proof proceeds from the formula

$$\phi_X(t) = \sum_k e^{ikt}P(X=k)\,.$$

Now suppose X has continuous bounded density f. Define

$$X_n = [nX]/n$$

where $[a]$ denotes the integer part (rounding down to the next smallest integer). We have

$$\begin{aligned} P(k/n \le X &< (k+1)/n) \\ &= P([nX] = k) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_{[nX]}(t) \times e^{-itk} dt\,. \end{aligned}$$

Make the substitution $t = u/n$, and get

$$\begin{aligned} nP(k/n \le X < (k+1)/n) = \frac{1}{2\pi} \\ \times \int_{-n\pi}^{n\pi} \phi_{[nX]}(u/n) e^{-iuk/n} du\,. \end{aligned}$$

Now, as $n \to \infty$ we have

$$\phi_{[nX]}(u/n) = E(e^{iu[nX]/n}) \to E(e^{iuX}) .$$

(Dominated convergence: $|e^{iu}| \le 1$.)

Range of integration converges to the whole real line.

If $k/n \to x$ left hand side converges to density $f(x)$ while right hand side converges to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(u)e^{-iux}du$$

which gives the inversion formula

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(u)e^{-iux}du .$$

Many other such formulas are available to compute things like $F(b) - F(a)$ and so on.

All such formulas called **Fourier inversion formulas**.

Characteristic ftn also called **Fourier transform** of f or F.

Inversion of the Moment Generating Function

MGF and characteristic function related formally:

$$M_X(it) = \phi_X(t)\,.$$

When M_X exists this relationship is not merely formal; the methods of complex variables mean there is a "nice" (analytic) function which is $E(e^{zX})$ for any complex $z = x + iy$ for which $M_X(x)$ is finite.

SO: there is an inversion formula for M_X using a complex *contour integral*:

If z_1 and z_2 are two points in the complex plane and C a path between these two points we can define the path integral

$$\int_C f(z)dz$$

by the methods of line integration.

Do algebra with such integrals via usual theorems of calculus.

The Fourier inversion formula was

$$2\pi f(x) = \int_{-\infty}^{\infty} \phi(t) e^{-itx} dt$$

so replacing ϕ by M we get

$$2\pi f(x) = \int_{-\infty}^{\infty} M(it) e^{-itx} dt \, .$$

If we just substitute $z = it$ then we find

$$2\pi i f(x) = \int_C M(z) e^{-zx} dz$$

where the path C is the imaginary axis.

Complex contour integration: replace C by any other path which starts and ends at the same place.

Sometimes can choose path to make it easy to do the integral approximately; this is what **saddlepoint approximations** are.

Inversion formula is called the **inverse Laplace transform**; the mgf is also called the Laplace transform of f or F.

Applications of Inversion

1): Numerical calculations

Example: Many statistics have a distribution which is approximately that of

$$T = \sum \lambda_j Z_j^2$$

where the Z_j are iid $N(0,1)$. In this case

$$\begin{aligned} E(e^{itT}) &= \prod E(e^{it\lambda_j Z_j^2}) \\ &= \prod (1-2it\lambda_j)^{-1/2}\,. \end{aligned}$$

Imhof (*Biometrika*, 1961) gives a simplification of the Fourier inversion formula for

$$F_T(x) - F_T(0)$$

which can be evaluated numerically:

$$\begin{aligned} &F_T(x) - F_T(0) \\ &\qquad = \int_0^x f_T(y)dy \\ &\qquad = \int_0^x \frac{1}{2\pi}\int_{-\infty}^{\infty} \prod (1-2it\lambda_j)^{-1/2} e^{-ity} dt dy\,. \end{aligned}$$

Multiply

$$\phi(t) = \left[\frac{1}{\prod(1-2it\lambda_j)}\right]^{1/2}$$

top and bottom by the complex conjugate of the denominator:

$$\phi(t) = \left[\frac{\prod(1+2it\lambda_j)}{\prod(1+4t^2\lambda_j^2)}\right]^{1/2}.$$

The complex number $1+2it\lambda_j$ is $r_j e^{i\theta_j}$ where

$$r_j = \sqrt{1+4t^4\lambda_j^2}$$

and

$$\tan(\theta_j) = 2t\lambda_j\,.$$

This allows us to rewrite

$$\phi(t) = \left[\frac{\prod r_j e^{i\sum \theta_j}}{\prod r_j^2}\right]^{1/2}$$

or

$$\phi(t) = \frac{e^{i\sum \tan^{-1}(2t\lambda_j)/2}}{\prod(1+4t^2\lambda_j^2)^{1/4}}.$$

Assemble this to give

$$F_T(x) - F_T(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\theta(t)}}{\rho(t)} \int_0^x e^{-iyt} dy dt$$

where

$$\theta(t) = \sum \tan^{-1}(2t\lambda_j)/2$$

and $\rho(t) = \prod(1+4t^2\lambda_j^2)^{1/4}$. But

$$\int_0^x e^{-iyt} dy = \frac{e^{-ixt} - 1}{-it}.$$

We can now collect up the real part of the resulting integral to derive the formula given by Imhof. I don't produce the details here.

2): The central limit theorem (in some versions) can be deduced from the Fourier inversion formula: if $X_1, \ldots, X_n$ are iid with mean 0 and variance 1 and $T = n^{1/2}\bar{X}$ then with ϕ denoting the characteristic function of a single X we have

$$
\begin{aligned}
E(e^{itT}) &= E(e^{in^{-1/2}t\sum X_j}) \\
&= \left[\phi(n^{-1/2}t)\right]^n \\
&\approx \left[\phi(0) + \frac{t\phi'(0)}{\sqrt{n}} + \frac{t^2\phi''(0)}{2n} + o(n^{-1})\right]^n
\end{aligned}
$$

But now $\phi(0) = 1$ and

$$\phi'(t) = \frac{d}{dt}E(e^{itX_1}) = iE(X_1 e^{itX_1}) \,.$$

So $\phi'(0) = E(X_1) = 0$. Similarly

$$\phi''(t) = i^2 E(X_1^2 e^{itX_1})$$

so that

$$\phi''(0) = -E(X_1^2) = -1 \,.$$

It now follows that

$$
\begin{aligned}
E(e^{itT}) &\approx [1 - t^2/(2n) + o(1/n)]^n \\
&\to e^{-t^2/2} \,.
\end{aligned}
$$

With care we can then apply the Fourier inversion formula and get

$$
\begin{aligned}
f_T(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}[\phi(tn^{-1/2})]^n dt \\
&\to \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} e^{-t^2/2} dt \\
&= \frac{1}{\sqrt{2\pi}}\phi_Z(-x)
\end{aligned}
$$

where ϕ_Z is the characteristic function of a standard normal variable Z. Doing the integral we find

$$\phi_Z(x) = \phi_Z(-x) = e^{-x^2/2}$$

so that

$$f_T(x) \to \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

which is a standard normal density.

Proof of the central limit theorem not general: requires T to have bounded continuous density.

Central limit theorem: statement about cdfs not densities:

$$P(T \le t) \to P(Z \le t)\,.$$

3) Saddlepoint approximation from MGF inversion formula

$$2\pi i f(x) = \int_{-i\infty}^{i\infty} M(z)e^{-zx}dz$$

(limits of integration indicate contour integral running up imaginary axis.)

Replace contour (using complex variables) with line $Re(z) = c$. ($Re(Z)$ denotes the real part of z, that is, x when $z = x + iy$ with x and y real.) Must choose c so that $M(c) < \infty$. Rewrite inversion formula using cumulant generating function $K(t) = \log(M(t))$:

$$2\pi i f(x) = \int_{c-i\infty}^{c+i\infty} \exp(K(z) - zx)dz\,.$$

Along the contour in question we have $z = c + iy$ so we can think of the integral as being

$$i\int_{-\infty}^{\infty} \exp(K(c+iy) - (c+iy)x)dy\,.$$

Now do a Taylor expansion of the exponent:

$$\begin{aligned}&K(c+iy) - (c+iy)x = \\ &\; K(c) - cx + iy(K'(c) - x) - y^2K''(c)/2 + \cdots\,.\end{aligned}$$

Ignore the higher order terms and select a c so that the first derivative

$$K'(c) - x$$

vanishes. Such a c is a saddlepoint. We get the formula

$$\begin{aligned}2\pi f(x) &\approx \exp(K(c) - cx) \\ &\qquad \times \int_{-\infty}^{\infty} \exp(-y^2K''(c)/2)dy\,.\end{aligned}$$

Integral is normal density calculation; gives

$$\sqrt{2\pi/K''(c)}\,.$$

Saddlepoint approximation is

$$f(x) = \frac{\exp(K(c) - cx)}{\sqrt{2\pi K''(c)}}\,.$$

Essentially same idea: Laplace's approximation.

Example: Sterling's approximation to factorial:

$$n! = \int_0^\infty \exp(n\log(x) - x)dx\,.$$

Exponent maximized when $x = n$.

For n large approximate $f(x) = n\log(x) - x$ by

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0) + (x - x_0)^2 f''(x_0)/2$$

and choose $x_0 = n$ to make $f'(x_0) = 0$. Then

$$n! \approx \int_0^\infty \exp[n\log(n) - n - (x-n)^2/(2n)]dx\,.$$

Substitute $y = (x-n)/\sqrt{n}$; get approximation

$$n! \approx n^{1/2} n^n e^{-n} \int_{-\infty}^\infty e^{-y^2/2} dy$$

or

$$n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n}\,.$$

Note: sloppy about limits of integration.

Real proof must show integral over x not near n is negligible.