0.0.1 Inversion

The previous theorem is non-constructive characterization. That is, it says that ϕ_X determines F_X and f_X but it does not say how to find the latter from the former. This raises the question: Can get from ϕ_X to F_X or f_X by **inversion**.

If X is a random variable taking only integer values then for each integer k

$$\begin{aligned} P(X=k) &= \frac{1}{2\pi}\int_0^{2\pi} \phi_X(t)e^{-itk}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_X(t)e^{-itk}dt\,. \end{aligned}$$

The proof proceeds from the formula

$$\phi_X(t) = \sum_k e^{ikt}P(X=k)\,.$$

You multiply this by e^{-ijt} and integrate from 0 to 2π. This produces

$$\int_0^{2\pi} e^{-ijt}\phi_X(t)\,dt = \sum_k P(X=k)\int_0^{2\pi} e^{i(k-j)t}\,dt.$$

Now for $k \neq j$ the derivative of

$$e^{i(k-j)t}$$

with respect to t is just

$$i(k-j)e^{i(k-j)t}$$

so the integral is simply

$$\left.\frac{e^{i(k-j)t}}{i(k-j)}\right|_{t=0}^{t=2\pi} = \frac{\cos(2(k-j)\pi)+i\sin(2(k-j)\pi)-\cos(0)-i\sin(0)}{i(k-j)} = \frac{1+0i-1-0i}{i(k-j)} = 0.$$

The integral with $k=j$, however, is different. It is just

$$\int_0^{2\pi} e^{i0t}dt = \int_0^{2\pi} 1dt = 2\pi.$$

So

$$\int_0^{2\pi} e^{-ijt}\phi_X(t)\,dt = 2\pi P(X=j).$$

Now suppose X has continuous bounded density f. Define

$$X_n = [nX]/n$$

where $[a]$ denotes the integer part (rounding down to the next smallest integer). We have

$$\begin{aligned}P(k/n \le X &< (k+1)/n)\\ &= P([nX]=k)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_{[nX]}(t)\times e^{-itk}dt\,.\end{aligned}$$

Make the substitution $t = u/n$, and get

$$nP(k/n \le X < (k+1)/n) = \frac{1}{2\pi}\times\int_{-n\pi}^{n\pi}\phi_{[nX]}(u/n)e^{-iuk/n}du\,.$$

Now, as $n\to\infty$ we have

$$\phi_{[nX]}(u/n) = E(e^{iu[nX]/n}) \to E(e^{iuX})\,.$$

(Dominated convergence: $|e^{iu}| \le 1$.)

Range of integration converges to the whole real line.

If $k/n \to x$ left hand side converges to density $f(x)$ while right hand side converges to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_X(u)e^{-iux}du$$

which gives the inversion formula

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_X(u)e^{-iux}du\,.$$

Many other such formulas are available to compute things like $F(b)-F(a)$ and so on; the book by Loève on probability is a good source for such formulas and their proofs.

All such formulas are called **Fourier inversion formulas**. The characteristic function is also called the **Fourier transform** of f or F.

0.0.2 Inversion of the Moment Generating Function and Saddlepoint Approximations

The moment generating function and the characteristic function are related formally:

$$M_X(it) = \phi_X(t)\,.$$

When M_X exists this relationship is not merely formal; the methods of complex variables mean there is a "nice" (analytic) function which is $E(e^{zX})$ for any complex $z = x + iy$ for which $M_X(x)$ is finite. So: there is an inversion formula for M_X using a complex *contour integral*:

If z_1 and z_2 are two points in the complex plane and C a path between these two points we can define the path integral

$$\int_C f(z)dz$$

by the methods of line integration.

The inversion formula just derived was

$$2\pi i f(x) = \int_{-\infty}^{\infty} M_X(it)e^{-itx}dt$$

Now imagine making a change of variables to $z = it$. As t, a real variable, goes from $-\infty$ to ∞ the variable z runs up the imaginary axis. We also have $dz = i\,dt$. This leads to the following inversion formula for the moment generating function

$$2\pi i f(x) = \int_{-i\infty}^{i\infty} M(z)e^{-zx}dz$$

(the limits of integration indicate a contour integral running up the imaginary axis.)

It is now possible to replace contour (using complex variables theory) with the line $Re(z) = c$. ($Re(Z)$ denotes the real part of z, that is, x when $z = x + iy$ with x and y real.) We must choose c so that $M(c) < \infty$. In this case we rewrite the inversion formula using the cumulant generating function $K(t) = \log(M(t))$ in the following form:

$$2\pi i f(x) = \int_{c-i\infty}^{c+i\infty} \exp(K(z) - zx)dz\,.$$

Along the contour in question we have $z = c + iy$ so we can think of the integral as being

$$i \int_{-\infty}^{\infty} \exp(K(c+iy) - (c+iy)x)dy\,.$$

Now we do a Taylor expansion of the exponent:

$$K(c+iy) - (c+iy)x = K(c) - cx + iy(K'(c) - x) - y^2 K''(c)/2 + \cdots\,.$$

Ignore the higher order terms and select a c so that the first derivative

$$K'(c) - x$$

vanishes. Such a c is called a *saddlepoint*. We get the formula

$$2\pi f(x) \approx \exp(K(c) - cx) \int_{-\infty}^{\infty} \exp(-y^2 K''(c)/2)dy\,.$$

The integral is a normal density calculation; it gives

$$\sqrt{2\pi/K''(c)}\,.$$

Thus our saddlepoint approximation is

$$f(x) \approx \frac{\exp(K(c) - cx)}{\sqrt{2\pi K''(c)}}\,.$$

The tactic used here is essentially the same idea as in Laplace's approximation whose most famous example is Stirling's formula

Example: Stirling's approximation to a factorial. We may show, by induction on n and integration by parts that

$$n! = \int_0^{\infty} \exp(n\log(x) - x)dx\,.$$

The exponent is maximized when $x = n$. For n large we approximate $f(x) = n\log(x) - x$ by

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0) + (x - x_0)^2 f''(x_0)/2$$

and choose $x_0 = n$ to make $f'(x_0) = 0$. Then

$$n! \approx \int_0^\infty \exp[n\log(n) - n - (x-n)^2/(2n)]dx\,.$$

Substitute $y = (x-n)/\sqrt{n}$; get approximation

$$n! \approx n^{1/2} n^n e^{-n} \int_{-\infty}^\infty e^{-y^2/2} dy$$

or

$$n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n}\,.$$

Note: I am being quite sloppy about limits of integration; this is a fixable error but I won't be doing the fixing. A real proof must show that the integral over x not near n is negligible.