Inversion of Generating Functions

Previous theorem is non-constructive characterization. Can get from $\phi_X$ to $F_X$ or $f_X$ by inversion. See homework for basic inversion formula:
If $X$ is a random variable taking only integer values then for each integer $k$

$$P(X = k) = \frac{1}{2\pi} \int_0^{2\pi} \phi_X(t)e^{-ikt}dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_X(t)e^{-ikt}dt.$$  

The proof proceeds from the formula

$$\phi_X(t) = \sum_k e^{ikt} P(X = k).$$

Now suppose that $X$ has a continuous bounded density $f$. Define

$$X_n = \lceil nX \rceil / n$$

where $\lceil a \rceil$ denotes the integer part (rounding down to the next smallest integer). We have

$$nP(k/n \leq X < (k + 1)/n) = P([nX] = k)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_{[nX]}(t)$$

$$\times e^{-ikt}dt.$$  

Make the substitution $t = u/n$, and get

$$nP(k/n \leq X < (k + 1)/n) = \frac{1}{2\pi} \times \int_{-n\pi}^{n\pi} \phi_{[nX]}(u/n)e^{iku/n}du$$

Now, as $n \to \infty$ we have

$$\phi_{[nX]}(u/n) = E(e^{iu[nX]/n}) \to E(e^{iuX})$$

(by the dominated convergence theorem – the dominating random variable is just the constant 1). The range of integration converges to the whole real line and if $k/n \to x$ we see that the left hand side converges to the density $f(x)$ while the right hand side converges to

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(u)e^{-iux}du$$

which gives the inversion formula

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(u)e^{-iux}du$$

Many other such formulas are available to compute things like $F(b) - F(a)$ and so on.

All such formulas are sometimes referred to as Fourier inversion formulas; the characteristic function itself is sometimes called the Fourier transform of the distribution or cdf or density of $X$.  

Inversion of the Moment Generating Function
MGF and characteristic function related formally:

\[ M_X(it) = \phi_X(t) \]

When \( M_X \) exists this relationship is not merely formal; the methods of complex variables mean there is a “nice” (analytic) function which is \( E(e^{zX}) \) for any complex \( z = x + iy \) for which \( M_X(x) \) is finite.

SO: there is an inversion formula for \( M_X \) using a complex contour integral:

If \( z_1 \) and \( z_2 \) are two points in the complex plane and \( C \) a path between these two points we can define the path integral

\[ \int_C f(z)dz \]

by the methods of line integration.

Do algebra with such integrals via usual theorems of calculus. The Fourier inversion formula was

\[ 2\pi f(x) = \int_{-\infty}^{\infty} \phi(t)e^{-itx}dt \]

so replacing \( \phi \) by \( M \) we get

\[ 2\pi f(x) = \int_{-\infty}^{\infty} M(it)e^{-itx}dt \]

If we just substitute \( z = it \) then we find

\[ 2\pi f(x) = \int_C M(z)e^{-\bar{z}x}dz \]

where the path \( C \) is the imaginary axis. Methods of complex integration permit us to replace \( C \) by any other path which starts and ends at the same place. Sometimes can choose path to make it easy to do the integral approximately; this is what saddlepoint approximations are. Inversion formula is called the inverse Laplace transform; the mgf is also called the Laplace transform of the distribution or cdf or density.

**Applications of Inversion**

1) Numerical calculations

Example: Many statistics have a distribution which is approximately that of

\[ T = \sum \lambda_j Z_j^2 \]

where the \( Z_j \) are iid \( N(0,1) \). In this case

\[ E(e^{itT}) = \prod E(e^{it\lambda_j Z_j^2}) = \prod (1 - 2it\lambda_j)^{-1/2}. \]

Imhof (Biometrika, 1961) gives a simplification of the Fourier inversion formula for

\[ F_T(x) - F_T(0) \]

which can be evaluated numerically:

\[ F_T(x) - F_T(0) = \int_0^x f_T(y)dy \]

\[ = \int_0^x \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod (1 - 2it\lambda_j)^{-1/2}e^{-ity}dtdy \]
Multiply
\[
\phi(t) = \left[ \frac{1}{\prod (1 - 2it\lambda_j)} \right]^{1/2}
\]
top and bottom by the complex conjugate of the denominator:
\[
\phi(t) = \left[ \frac{\prod (1 + 2it\lambda_j)}{\prod (1 + 4t^2\lambda_j^2)} \right]^{1/2}
\]
The complex number \(1 + 2it\lambda_j\) is \(r_je^{i\theta_j}\) where \(r_j = \sqrt{1 + 4t^2\lambda_j^2}\) and \(\tan(\theta_j) = 2t\lambda_j\). This allows us to rewrite
\[
\phi(t) = \left[ \frac{\prod r_je^{i\sum \theta_j}}{\prod r_j^2} \right]^{1/2}
\]
or
\[
\phi(t) = \frac{e^{i\sum \tan^{-1}(2t\lambda_j)/2}}{\prod (1 + 4t^2\lambda_j^2)^{1/4}}
\]
Assemble this to give
\[
F_T(x) - F_T(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{x} e^{-yit} dy dt
\]
where
\[
\theta(t) = \sum \tan^{-1}(2t\lambda_j)/2
\]
and \(\rho(t) = \prod (1 + 4t^2\lambda_j^2)^{1/4}\). But
\[
\int_{0}^{x} e^{-yit} dy = \frac{e^{-i\pi t} - 1}{-it}
\]
We can now collect up the real part of the resulting integral to derive the formula given by Imhof. I don’t produce the details here.

2): The central limit theorem (in some versions) can be deduced from the Fourier inversion formula: if \(X_1, \ldots, X_n\) are iid with mean 0 and variance 1 and \(T = n^{1/2}X\) then with \(\phi\) denoting the characteristic function of a single \(X\) we have
\[
E(e^{itX}) = E(e^{in^{-1/2}i\sum X_i}) = \phi(n^{-1/2}t)^n \\
\approx \phi(0) + \frac{t\phi'(0)}{\sqrt{n}} + \frac{t^2\phi''(0)}{2n} + o(n^{-1})
\]
But now \(\phi(0) = 1\) and
\[
\phi'(t) = \frac{d}{dt}E(e^{itX_1}) = iE(X_1e^{itX_1})
\]
So \(\phi'(0) = E(X_1) = 0\). Similarly
\[
\phi''(t) = i^2E(X_1^2e^{itX_1})
\]
so that
\[
\phi''(0) = -E(X_1^2) = -1
\]
It now follows that
\[
E(e^{itX}) \approx [1 - t^2/(2n) + o(1/n)]^n \to e^{-t^2/2}.
\]
With care we can then apply the Fourier inversion formula and get

\[ f_T(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} [\phi(t)]^n dt \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-t^2/2} dt \]

\[ = \frac{1}{\sqrt{2\pi}} \phi_Z(-x) \]

where \( \phi_Z \) is the characteristic function of a standard normal variable \( Z \). Doing the integral we find

\[ \phi_Z(x) = \phi_Z(-x) = e^{-x^2/2} \]

so that

\[ f_T(x) \to \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \]

which is a standard normal random variable.

This proof of the central limit theorem is not terribly general since it requires \( T \) to have a bounded continuous density. The central limit theorem itself is a statement about cdfs not densities and is

\[ P(T \leq t) \to P(Z \leq t) . \]

3) Saddlepoint approximation from MGF inversion formula

\[ 2\pi if(x) = \int_{-i\infty}^{i\infty} M(z)e^{-zx} dz \]

(limits of integration indicate contour integral running up imaginary axis.) Replace contour (using complex variables) with line \( Re(z) = c \). (\( Re(Z) \) denotes the real part of \( z \), that is, \( x \) when \( z = x + iy \) with \( x \) and \( y \) real.) Must choose \( c \) so that \( M(c) < \infty \). Rewrite inversion formula using cumulant generating function \( K(t) = \log(M(t)) \):

\[ 2\pi if(x) = \int_{c-i\infty}^{c+i\infty} \exp(K(z) - zx)dz . \]

Along the contour in question we have \( z = c + iy \) so we can think of the integral as being

\[ i \int_{-\infty}^{\infty} \exp(K(c + iy) - (c + iy)x)dy \]

Now do a Taylor expansion of the exponent:

\[ K(c + iy) - (c + iy)x = K(c) - cx + iy(K'(c) - x) - y^2K''(c)/2 + \cdots \]

Ignore the higher order terms and select a \( c \) so that the first derivative

\[ K'(c) - x \]

vanishes. Such a \( c \) is a saddlepoint. We get the formula

\[ 2\pi f(x) \approx \exp(K(c) - cx) \times \int_{-\infty}^{\infty} \exp(-y^2K''(c)/2)dy . \]

The integral is just a normal density calculation and gives \( \sqrt{2\pi/K''(c)} \). The saddlepoint approximation is

\[ f(x) = \frac{\exp(K(c) - cx)}{\sqrt{2\pi K''(c)}} . \]
Essentially the same idea lies at the heart of the proof of Sterling’s approximation to the factorial function:

\[ n! = \int_0^\infty \exp(n \log(x) - x)dx \]

The exponent is maximized when \( x = n \). For \( n \) large we approximate \( f(x) = n \log(x) - x \) by

\[ f(x) \approx f(x_0) + (x - x_0)f'(x_0) + (x - x_0)^2 f''(x_0)/2 \]

and choose \( x_0 = n \) to make \( f'(x_0) = 0 \). Then

\[ n! \approx \int_0^\infty \exp[n \log(n) - n - (x - n)^2/(2n)]dx \]

Substitute \( y = (x - n)/\sqrt{n} \) to get the approximation

\[ n! \approx n^{1/2}n^n e^{-n} \int_{-\infty}^\infty e^{-y^2/2}dy \]

or

\[ n! \approx \sqrt{2\pi/n} n^{n+1/2}e^{-n} \]

This tactic is called Laplace’s method. Note that I am being very sloppy about the limits of integration; to do the thing properly you have to prove that the integral over \( x \) not near \( n \) is negligible.