STAT 450

Problems: Assignment 4

1. Suppose that $X_1, \ldots, X_n$ are independent Poisson(1) random variables. Show that $n^{1/2}\log(\bar{X}_n)$ converges in distribution to $N(0, \tau^2)$ and compute τ.

2. Suppose $X_1, \ldots, X_n$ are independent Exponential(1) random variables. Let $T_n = \bar{X}_n$. Accordinding to the central limit theorem

$$P(n^{1/2}(T_n - 1) \le x) \to \int_{-\infty}^{x} \frac{1}{2\pi} e^{-u^2/2} du$$

In this problem I want you to **prove** that the δ method works in a special case.

(a) Let $g(x) = \log(x)$ and $U_n = \log(T_n) = g(T_n)$. Let

$$F_n(x) = P(n^{1/2}(T_n - 1) \le x)$$

and

$$G_n(u) = P(n^{1/2}U_n \le u)$$

Express $G_n(u)$ in terms of F_n; that is write

$$G_n(u) = F_n(\text{formula in } u).$$

(b) Use the previous part to compute

$$\lim_{n\to\infty} G_n(u)$$

(c) Show that the limit in part b) is the same as predicted by the δ method.

3. Consider the previous problem but now assume the data are iid exponential with mean μ. In this problem I want you to compare the coverage probabilities of 4 confidence intervals for μ:

(a) The interval based on the t_{n-1} approximation to the t statistic.

(b) The interval based on the pivot

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\mu}.$$

(c) The interval based on the pivot

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\bar{X}}.$$

(d) The interval based on the pivot

$$\sqrt{n}(\log(\bar{X}_n) - \log(\mu)).$$

I want you to plot the coverage probabilities of a 95% interval against n for $n = 5, 10, 25, 50, 100, 200$. Put them all on the same graph and add a horizontal line at 95%. The quantity T_n has a Gamma distribution with shape n and scale μ/n. You can use the function `pgamma` in `R` to compute the cdf of T_n and this allows you to compute the exact coverage probabilities for the lst 3 intervals. For the t interval you should do a Monte Carlo study: for each sample size generate 10,000 samples of that size from the $\mu = 1$ distribution, work out the t intervals, see if they include 1 and compute the fraction covered.

4. Find the Fisher information for problems 3, 4, and 5 on the previous assignment if that makes sense.

5. For problems 3b and 3d evaluate $\mathrm{Var}_\theta(\ell'(\theta))$ and $-\mathrm{E}_\theta(\ell''(\theta))$ to see if these are equal.

6. Suppose we have a sample of size n_1 from a $N(\mu_1, \sigma_1^2)$ distribution and an independent sample of size n_2 from a $N(\mu_2, \sigma_2^2)$ distribution. Find the mle of $\mu_1 - \mu_2$. If the value of $n = n_1 + n_2$ is fixed what values of n_1, n_2 minimize the variance of the mle of $\mu_1 - \mu_2$.

7. Suppose we measure the radius of a circle n times. Each measurement has an independent $N(0, \sigma^2)$ error. What is the mle of the area of the circle? What is wrong with the assumed model?