Name: ________________ **Student Number**: ________________

STAT 380: Spring 2016

Midterm Examination

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Instructions: This is a closed book exam. You are permitted to use 2 sheets of notes, machine-written or hand-written. You may use both sides of the sheets and I place no limits on font size. Calculators are not permitted nor are any other electronic aids. The exam is out of 25. Please put your name on each page. You should have 8 pages; the first page is a grade sheet and the last is extra space. I will be marking for clarity of explanation as well as correctness. Without a clear explanation you should not expect to get more than half marks.

1. A Markov Chain has state space $\{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Identify all the communicating classes, say whether or not each is transient, and give the period of each state. [5 marks]

Evidently 1 leads to 2 and to 5.

Since both 2 and 5 lead to 1 all three must be in the same communicating class. Many people seemed to feel that 5 leads only to 5. They are misleading the 1 in the 5th row.

When you are in any of states 1, 2, or 5 there is no chance you go to 3, 4, or 6 so that $\{1, 2, 5\}$ is a communicating class.

Next: state 3 leads to 4 and 4 leads to 3 so these two must be in the same communicating class. A lot of students got this wrong.

Finally 6 does not lead to any states other than 6 so it must be in a class of its own.

There are 3 communicating classes: $\{1, 2, 5\}$, $\{3, 4\}$, and $\{6\}$.

The first and last of these classes are recurrent and but the class $\{3, 4\}$ is transient since there is a positive chance of going either to 1 or 5 or 6 and then you can never get back.

The first and third classes are aperiodic while the class $\{3, 4\}$ has period 2. Notice that $P_{11} > 0$ so that the period of state 1 must divide the number 1!. That makes state 1 have period 1 so 2 and 5 must have period 1. Similarly state 6 has period 1. Now think about 3 and 4. What happens when you multiply P by itself is that entries $3, 4$ and $4, 3$ become 0. Take row 3 and dot product that with column 4. You get 0! Same for row 3 and column 4. The entries $3, 3$ and $4, 4$ are not 0; they are $1/9$. If you think about more multiplication you will see that these 4 elements just alternate. For even powers P^n has $3, 3$ and $4, 4$ entries which are positive; they are powers of $1/3$. For odd powers those entries are 0. That makes the relevant greatest common divisor 2.

2. There is an experimental design strategy called play-the-winner. A simplified version goes like this. Imagine two players, A and B, play a game. On each turn of the game one player 'serves' and can either score a point on that turn or not. If the player who served scores a point she serves again. If not, no point is scored and the other player begins to serve. Suppose that when A serves she scores a point with probability p_A and that when B serves she scores a point with probability p_B.

   (a) Define a suitable Markov chain to analyse this system. [4 marks]

   The simplest chain is to define X_n to be the player serving on turn n (in which case there is no X_0 or you can define the first serve to be $n = 0$ and so on. Thus the chain has states $\{A, B\}$. A certain number of people simply did not say "Let X_n be ..."; you have to do that step.
   It is also possible, but harder, to do this with a 4 state Markov Chain where at time n you record both who serves and the outcome of the service. No one who used this method succeeded with following it through all the way.

   (b) Write out the transition matrix of the resulting Markov Chain. [3 marks]

   The transition matrix is

$$P = \begin{bmatrix} p_A & 1 - p_A \\ 1 - p_B & p_B \end{bmatrix}$$

The most common mistake was to switch the two entries in the second row. In that case I tried not to penalize you on the rest of the questions after docking marks on this part.

(c) Assume that a fair coin is tossed to see who serves first. What is the probability that A serves on the third turn? [3 marks]

The initial distribution is

$$\alpha = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \end{array} \right]$$

and we are asked for

$$(\alpha P^2)_A$$

Now

$$P^2 = \left[\begin{array}{cc} p_A^2 + (1-p_A)(1-p_B) & p_A(1-p_A) + (1-p_A)p_B \\ p_A(1-p_B) + p_B(1-p_B) & p_B^2 + (1-p_B)(1-p_A) \end{array} \right]$$

so

$$(\alpha P^2)_A = \frac{p_A^2 + (1-p_A)(1-p_B) + p_A(1-p_B) + p_B(1-p_B)}{2} = \frac{p_A^2 + 1 - p_B^2}{2}.$$

If you number turns starting at 0 to match the notation standard notation in this course then the question asks for $P(X_2 = A)$ not $P(X_3 = A)$. I gave 2/3 for people who did the harder, but wrong, calculation.

(d) In the long run on what fraction of the turns does A serve? [4 marks]

The stationary initial distribution solves

$$\alpha = \alpha P.$$

The first of these two equations is

$$\alpha_A = \alpha_A p_A + \alpha_B(1-p_B).$$

Substitute $\alpha_B = 1 - \alpha_A$ to get

$$(1 - p_A + 1 - p_B)\alpha_A = 1 - p_B$$

so

$$\alpha_A = \frac{1-p_B}{2-p_A-p_B}.$$

This is the long run fraction of turns on which A serves.
Some people computed the stationary initial distribution but did not answer the question asked!

(e) Also in the long run what is the average number of points scored per turn (by either player)? [3 marks]

In the long run the fraction of time you are in state A is α_A from the previous part. When you are in state A you score a point with probability p_A. You are in state B $\alpha_B = 1 - \alpha_A$ of the time and a point is then scored with probability p_B. The desired long run fraction of turns where a point is scored is

$$\alpha_A p_A + \alpha_B p_B = \frac{p_A(1-p_B) + p_B(1-p_A)}{2 - p_A - p_B}.$$

I got a lot of answers which were the same as the previous part as if points were scored on every turn.

(f) Let M_n be the number of times that the serve changes in the first n turns. (For clarity if the player serving on turn n does not score a point that counts as a change of serve in the first n trials.) Let $\mu_{A,n} = \mathrm{E}(M_n)$ given that A serves first. Let $\mu_{B,n}$ be the same expected value given that B serves first. Use first step analysis to derive equations for $\mu_{A,n}$ and $\mu_{B,n}$ in terms of $\mu_{A,n-1}$ and $\mu_{B,n-1}$. Find the values for $n = 1$. Do not solve the set of equations. [3 marks]

If you start in A then on the first turn either A scores a point in which case you have seen 0 serve changes. You are now starting with A serving and counting serve changes in the next $n-1$ turns. If A does not score a point then you have had one service change and are now starting in on $n-1$ turns with B serving first. So

$$\mu_{A,n} = p_A(0+\mu_{A,n-1}) + (1-p_A)(1+\mu_{B,n-1}) = p_A\mu_{A,n-1} + (1-p_A)(1+\mu_{B,n-1}).$$

Similarly

$$\mu_{B,n} = p_B(0+\mu_{B,n-1}) + (1-p_B)(1+\mu_{A,n-1}) = p_B\mu_{B,n-1} + (1-p_B)(1+\mu_{A,n-1}).$$

Evidently

$$\mu_{A,1} = 1 - p_A$$

and

$$\mu_{B,n-1} = 1 - p_B.$$

Marks were pretty low on this problem.