STAT 380

Assignment 5

1. Imagine that while a spacecraft is in orbit it is hit by micrometeorites at the times of a Poisson process with a rate of λ per second. If the spaceship is in orbit for T seconds what is the chance that it is not hit by any micrometeorites.

   This is just the chance that $N(s+T) - N(s) = 0$ where $N(t)$ is the number of hits during the interval from 0 to time t and s is the time the spacecraft starts in orbit. So the desired probability comes from the Poisson(λT) distribution:

   $$\exp\{-\lambda T\}.$$

2. You and a friend arrive at a customer service counter to find there are two lines; each line has 1 person in it. Line 1 is being served by someone who takes an exponentially distributed amount of time to serve a customer with rate λ_1 per unit time. For Line 2 the service times have an exponential distribution with rate λ_2. All the various service times are independent. If you join Line 1 and your friend joins Line 2 what is the chance that you will start being served before your friend starts being served and what is the chance that you will finish being served before you friend?

   Let T_1 be the time required for the person being served in Line 1 to finish and T_2 be time required for the person being served in Line 2 to finish. Then let U_1 and U_2 be the service times for you and your friend. Let A_1 be the event that $T_1 < T_2$ and $A_2 = \{T_1 > T_2\}$. You will finish before your friend if one of the following happens:

   (a) A_1 happens. Then you finish before the person being served in line 2. Call this second event B_1.

   (b) A_1 happens. Then the person being served in line 2 finishes before you; call this B_2. So your friend starts service at that time. Then you finish before your friend; let C_2 be this event.

(c) A_2 happens. Then the person ahead of you in Line 1 finishes before your friend; call this B_3. Then you finish before your friend; call this C_3.

The point now is that each time the situation changes the lack of memory property can be used.

I will show:

$$P(A_1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Then I will show

$$P(B_1|A_1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Then I will show

$$P(B_2|A_1) = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

Next

$$P(C_2|A_1, B_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Next

$$P(A_2) = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

Next

$$P(B_3|A_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Finally

$$P(C_3|A_2, B_3) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

So the probability of the first way for the desired event to happen is

$$\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^2;$$

and the probability of the second way is

$$\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)\left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)$$

while the probability of the third way is

$$\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

Add these three together to get

$$\frac{\lambda_1^2\left(\lambda_1+3\lambda_2\right)}{\left(\lambda_1+\lambda_2\right)^3}.$$

Now why are all these formulas right? Each of these probabilities is the chance that the person being served in one line finishes before the person being served in the other. The lack of memory property means that each time someone finishes service a new race starts and you are just waiting to see which of two independent exponential variables is smaller. So we just need to compute

$$P(A_1) = P(T_1 < T_2).$$

There are lots of ways to do this. For instance:

$$P(T_2 > T_1 | T_1 = t) = e^{-\lambda_2 t}$$

because given $T_1 = t$ the variable T_2 has an exponential distribution and we are asked for its survival function. So

$$P(T_2 > T_1) = \mathrm{E}[P(T_2 > T_1 | T_1)] = \mathrm{E}(\exp\{-\lambda_2 T_2\}].$$

Now compute this as

$$\int_0^\infty e^{-\lambda_2 t}\lambda_1 e^{-\lambda_1 t}\,dt = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

3. With the two lines functioning as in the previous question imagine that you arrive (alone this time) to find line one empty and 1 person in line 2. If you know the values of λ_1 and λ_2 which line should you join? The answer may depend on the relation between the two λs.

If you join line 1 you will take an exponential amount of time with mean $1/\lambda_1$ while if you join line 2 you will take an exponential amount of time with mean $1/\lambda_2$ (the time for the person ahead of you to be served) plus another independent exponential amount of time with the same mean. So your expected service time is either $1/\lambda_1$ or $2/\lambda_2$.

The question does not say how to decide which line is better. If $2/\lambda_2 < 1/\lambda_1$ then on average you are better to go to Line 2 if $\lambda_2 > 2\lambda_1$, that is, the server in line 2 is more than twice as fast as the one in line 1. But what if you want to pick the line that probably finishes first. So let's let T_1 be the time it will take you if you join line 1 and $T_2 = T_2, 1 + T_2, 2$ be the time it will take you if you join line 2. I want to join line 1 if $P(T_1 < T_2) > 1/2$. This chance is

$$E[P(T_2 > T_1|T_1)].$$

The survival function of T_2 is

$$(1 + \lambda_2 t)e^{-\lambda_2 t}$$

(You can get this from the convolution of 2 exponentials.) So the chance is

$$\int_0^\infty (1 + \lambda_2 t)e^{-\lambda_2 t}\lambda_1 e^{-\lambda_1 t}\,dt = \frac{\lambda_1(\lambda_1 + 2\lambda_2)}{(\lambda_1 + \lambda_2)^2}.$$

It is possible to show that this is more than 1/2 if

$$\lambda_2 < (1 + \sqrt{2})\lambda_1.$$

Other ways of deciding what is best are possible. In marking I am okay with any way you used.

4. Page 345, number 46. A Poisson process N is independent of a random time $T \geq 0$. Assume T has mean μ and sd σ. Part a) find $\mathrm{Cov}(T, N(T))$. Part b) find $\mathrm{Var}(N(T))$.

Given $T = t$ $N(T)$ has a Poisson(λt) distribution so

$$\mathrm{E}(N(T)|T) = \lambda T$$

and

$$\text{Var}(N(T)|T) = \lambda T$$

Then

$$\begin{aligned}
\text{E}(TN(T)) &= \text{E}(\text{E}(TN(T)|T)) \\
&= \text{E}(T\text{E}(N(T)|T)) \\
&= \text{E}(T\lambda T) \\
&= \lambda(\text{Var}(T) + (\text{E}(T))^2) \\
&= \lambda(\sigma^2 + \mu^2)
\end{aligned}$$

Subtracting $\text{E}(T)\text{E}(N(T))$ and noting $\text{E}(N(T)) = \lambda\text{E}(T) = \lambda\mu$ gives

$$\text{Cov}(T, N(T)) = \lambda\sigma^2$$

For the variance we have

$$\begin{aligned}
\text{Var}(N(T)) &= \text{E}(\text{Var}(N(T)|T)) + \text{Var}(\text{E}(N(T)|T)) \\
&= \text{E}(\lambda T) + \text{Var}(\lambda T) \\
&= \lambda\mu + \lambda^2\sigma^2
\end{aligned}$$

5. Suppose that for each n X_n and Y_n are independent Binomial random variables. Assume that the distribution of X_n is Binomial(n, p_n) and the distribution of Y_n is Binomial(n, q_n). Assume that $p_n = \lambda/n$ and $q_n = \theta/n$. Let $W_n = X_n + Y_n$. Compute the probability generating function of W_n, namely $\phi_n(s) = \text{E}(s^{W_n})$. Then compute

$$\lim_{n\to\infty} \phi_n(s).$$

Compare the result to the probability generating function of a Poisson random variable and interpret the result.

The pgf of a Bernoulli(p) variable is

$$ps^1 + (1-p)s^0 = 1 - p + ps = 1 + p(s-1)$$

So the pgf of a Binomial(n, p) is

$$\{1 + p(s-1)\}^n$$

The pdf of W_n is

$$\left(1 + \frac{\lambda}{n}(s-1)\right)^n \left(1 + \frac{\theta}{n}(s-1)\right)^n$$

If we let $n \to \infty$ this converges to

$$e\lambda(s-1)e^{\theta(s-1)} = e^{(\lambda+\theta)(s-1)}$$

which is the pdf of a Poisson random variable with parameter $\lambda + \theta$. So the sum of two independent Poisson variables is Poisson and the parameters add up.

6. Page 346 number 49. For a rate λ Poisson process we try to stop at the last event before some fixed time T. Our strategy will be to stop at the first event after some time s. We lose if there are no more events after s and before T or if the next event after s is not the last event before T. What is the probability of winning for s fixed? What value of s maximizes this probability? What is the maximum value?

    For a) the probability of winning is $P(N(s,T] = 1) = \lambda(T - s)e^{-\lambda(T-s)}$

    For b) take d/ds and set equal to 0 to get $\lambda(T - s) = 1$ or $s = T - 1/\lambda$. The second derivative evaluated at this s is $-\lambda^2 e^{-1} < 0$ so this is a maximum.

    For c) plug in to get e^{-1}.