STAT 380 Notes on Continuous Time Markov Chains

I do not have detailed notes on Continuous Time Chains in this format.

As motivation for this section I will describe the Bienaymé Galton Watson process again and then talk about Branching Processes – a continuous time version.

BGW processes: I described this in terms of survival of family names. Traditionally in a number of cultures the family name follows sons. Consider a man at the end of the 20th century. What is the probability that there is a descendant (male) with same last name alive at end of 21st century or at the end of 30th century? Rather than answer this question I looked at *generations* of men, not years. I computed the probability that the last name would survive for n generations. It is technically easier to compute q_n, probability of extinction by generation n.

I introduced the following random variables:

$$X = \#\text{ of male children of first man}$$

and

$$Z_k = \#\text{ of male children in generation } k$$

The event of interest is

$$E_n = \{Z_n = 0\}$$

We want to compute $q_n = P(E_n)$. To do so we used first step analysis which means we conditioned on the value of $Z_1 = X$ and broke up E_n:

$$q_n = P(E_n) = \sum_{k=0}^{\infty} P(E_n \cap \{X = k\})$$

Now look at the event $E_n \cap \{X = k\}$. Let

$$\begin{aligned} B_{j,n-1} = &\{X = k\} \cap \{\text{child } j\text{s line extinct} \\ &\text{in } n-1 \text{ generations}\} \end{aligned}$$

Then the first man's line is extinct in n generations if and only if every one of his sons' lines is extinct in $n-1$ generations so:

$$E_n \cap \{X = k\} = \cap_{j=1}^{k} B_{j,n-1}$$

Now we add modelling assumptions to make Z_n a Markov Chain with stationary transition probabilities:

1. Given (*conditional on*) $X = k$ the events $B_{j,n-1}$ are independent. In other words: one son's descendants don't affect other sons' descendants.

2. Given $X = k$ the probability of $B_{j,n-1}$ is q_{n-1}. In other words: sons are just like the parent.

Now add the notation $P(X = k) = p_k$ and compute using probability generating functions:

$$\begin{aligned}
q_n &= \sum_{k=0}^{\infty} P(E_n \cap \{X = k\}) \\
&= \sum_{k=0}^{\infty} P(\cap_{j=1}^{k} B_{j,n-1} | X = k) p_k \\
&= \sum_{k=0}^{\infty} \prod_{1}^{k} P(B_{j,n-1} | X = k) p_k \\
&= \sum_{k=0}^{\infty} (q_{n-1})^k p_k
\end{aligned}$$

Continuous Time

In the continuous time version we replace Z_k by $Z(t)$, the population size at time t with $t \geq 0$. We start with $Z(0) = 1$ and we imagine that the population sizes evolve as follows: we wait a random amount of time and then the single parent is replaced by a random number X of offspring. Then each of these offspring behaves like an independent copy of the original parent: waiting a random time then being replaced by a random number of offspring.

Our version of this will be Markov and we specify the situation as follows: suppose that at time t there are n individuals. Then between time t and time $t + h$ there is a chance $np_k h + o(h)$ that one of n individuals is replaced by k individuals where *Hope I write more*