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Tests of Fit based on Normalized Spacings

By R. A. LOCKHART, F. J. O'REILLY and M. A. STEPHENS†

Simon Fraser University, Canada and IIMAS, UNAM, Mexico

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SUMMARY

Normalized spacings provide useful tests of fit for many suitably regular continuous distributions; attractive features of the tests are that they can be used with unknown parameters and also with samples which are censored (Type 2) on the left and/or right. A transformation of the spacings leads, under the null hypothesis, to a set of z-values in (0, 1); however, these are not uniformly distributed except for spacings from the exponential or uniform distributions. Statistics based on the mean or the median of the z-values have already been suggested for tests for the Weibull (or equivalently the extreme-value) distribution; we now add the Anderson–Darling statistic. Asymptotic theory of the test statistics is given in general, and specialized to the normal, logistic and extreme-value distributions. Monte Carlo results show the asymptotic points can be used for relatively small samples. Also, a Monte Carlo study on power of the normal tests is given, which shows the Anderson–Darling statistic to be powerful against a wide range of alternatives; the mean and median can be non-consistent or even biased.

Keywords: EXTREME-VALUE DISTRIBUTION; GOODNESS-OF-FIT; LEAPS; LOGISTIC DISTRIBUTION; NORMAL DISTRIBUTION; SPACINGS; TESTS OF DISTRIBUTIONAL ASSUMPTIONS; TESTS OF NORMALITY; WEIBULL DISTRIBUTION

1. INTRODUCTION: NORMALIZED SPACINGS

Normalized spacings

Suppose $F(w)$ is a completely specified continuous distribution function, and let x have a distribution function $G(x)$, where $x = \alpha + \beta w$; thus α and β are location and scale parameters in $G(x)$. If $x_{(1)} < x_{(2)} < \ldots < x_{(n)}$ are the order statistics of a random sample from $G(x)$, the $x_{(i)}$ can be represented as $x_{(i)} = \alpha + \beta w_{(i)}$ where w_i is a random sample from $F(w)$. Suppose $m_i = E(w_{(i)})$ where E denotes expectation; since $F(w)$ is completely specified, m_i can be calculated. The *spacings* between the $x_{(i)}$ are defined by $s_i = x_{(i)} - x_{(i-1)}$, $i = 2, 3, \ldots, n$ and the *normalized spacings* are $s_i/(m_i - m_{i-1})$, $i = 2, 3, \ldots, n$. Normalised spacings can be used to test whether the sample does indeed come from $G(x)$. This can be done for samples censored at either or both ends, as well as for complete samples, so we will describe the censored case. Suppose then the available observations are $x_{(k)}, x_{(k+1)}, \ldots, x_{(k+1+r)}$, and define the normalized spacings

$$y_i = \{x_{(k+i)} - x_{(k+i-1)}\}/(m_{k+i} - m_{k+i-1}), \quad i = 1, \ldots, r+1. \tag{1}$$

A further transformation J gives ordered values $z_{(i)}$ as follows.
Define $T_j = \Sigma y_i$, $i = 1, \ldots, j$, so that T_{r+1} is the sum of all the y_i. Then let

$$z_{(i)} = T_i/T_{r+1}, \quad i = 1, \ldots, r. \tag{2}$$

The values $z_{(i)}$ are clearly between 0 and 1.

Example: *The exponential distribution*

When the x-sample is from an exponential distribution $G_E(x) = 1 - \exp(-x/\beta)$, $x > 0$, the quantity $m_{k+i} - m_{k+i-1}$ in the denominator of y_i becomes $1/(n - k - i + 1)$, and it is

† *Address for correspondence*: Professor M. A. Stephens, Department of Mathematics and Statistics, Simon Fraser University, Burnaby, B.C., V5A IS6, Canada.

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well-known that the y_i are themselves a random sample from $G_E(\cdot)$ above. Furthermore, for y_i from $G_E(\cdot)$, the J transformation gives $z_{(i)}$, $i = 1, \ldots, r$ (note that $z_{(r+1)} \equiv 1$), which are distributed like *ordered uniforms*, that is, like the ordered values of a random sample of size r from the uniform distribution between 0 and 1, written $U(0, 1)$. (If the exponential sample is not censored on the left there is an extra normalized spacing derived from $x_{(1)}$ alone. This is $y_1 = x_{(1)}/m_1 = nx_{(1)}$; the other normalized spacings are obtained by allowing y_{i+1} to be the right hand side of (1), with $k = 1$, $x_{(0)} \equiv 0$, giving values $y_1, y_2, \ldots, y_{r+2}$; T and (2) are modified in an obvious way, to give one extra $z_{(i)}$.) In reliability theory, if the original $x_{(i)}$ were lifetimes of tested parts all of which were put on test at the same moment $x = 0$, the quantity T_i is often called the *total time on test* to failure i, since it may be shown to be $T_i = x_{(1)} + x_{(2)} + \ldots + x_{(i)} + (n - i)x_{(i)}$, that is, T_i is the sum of the times to failure of the first i items to fail, *plus* the time so far on test of the parts still working. Many test procedures are based on the values T_i. They can be shown to be related to the $z_{(i)}$, and the test for exponentiality of the x-sample has been converted to a test for uniformity of the z-sample. Tests of this type are discussed in Stephens (1986).

For a more general $G(x)$, the normalized spacings are *not* a random exponential sample, even asymptotically. An important theorem exists concerning the asymptotic properties of normalized spacings. This is that, as $n \to \infty$, for any regular parent population $G(x)$ for x, and for "sufficiently separate" indices k and l, y_k and y_l converge to independent exponentials, as k, l, $n \to \infty$, and $k/n \to p$ and $ln \to q$, with both p and q in (0, 1) and $p \neq q$; see Pyke (1965, p. 407) for more rigour and details.

This result might suggest that, as $n \to \infty$, the $z_{(i)}$ can be regarded as ordered uniforms, but this is not so; the condition that p cannot be 0 and q cannot be 1 and that they must be different prevents this result. Thus distribution theory of tests based on the $z_{(i)}$ must not be derived on this assumption. In this article we give the correct asymptotic theory of three such tests, in some generality, and then particularise it to the normal, the logistic, and the extreme-value distributions. Finally the normal tests are compared with other tests for power. Two of the three statistics, which have already been proposed in the literature, are found to be not always consistent and sometimes biased; the third statistic (A^2 below) gives good results and is the recommended statistic in this class. Percentage points and power results for tests for the extreme-value distribution (or equivalently, for the two-parameter Weibull distribution) are given elsewhere (Lockhart, O'Reilly and Stephens, 1986).

2. TEST STATISTICS AND ASYMPTOTIC THEORY

We investigate tests based on three statistics. The first one is the Anderson-Darling statistic A^2, calculated from the r values $z_{(i)}$ given by (2) by

$$A^2 = -r - (1/r)\left(\sum_{i=1}^{r} (2i - 1)[\log z_{(i)} + \log\{1 - z_{(r+1-i)}\}]\right); \tag{3}$$

here log x refers to natural logarithms. The other test statistics are Z_1 and Z_2 given by

$$\begin{aligned} Z_1 &= r^{1/2}[z_{(\{r+1\}/2)} - \tfrac{1}{2}], r \quad \text{odd} \\ &= r^{1/2}[z_{(\{r+2\}/2)} - (r + 2)/\{2(r + 1)\}], r \quad \text{even} \end{aligned} \tag{4}$$

$$Z_2 = r^{1/2}(\bar{z} - \tfrac{1}{2}) \text{ where } \bar{z} = \sum_{j=1}^{r} z_{(j)}/r. \tag{5}$$

Z_1 is derived from the median of the $z_{(i)}$ when r is odd and, when r is even, from the order statistic $z_{(\{r+2\}/2)}$, which is close to the median. Statistic Z_2 is derived from the mean $\bar{z}$ of the $z_{(i)}$. These statistics are investigated because they are closely related to statistic S, introduced by Mann, Scheuer and Fertig (1973) and statistic S^*, introduced by Tiku and Singh (1981), for tests for the extreme-value and Weibull distributions. When the x set is from an extreme-value distribution the statistic S is the same as $1 - z_{(t)}$, where $t = (r+1)/2$ when r is odd and $t = (r+2)/2$ when r is even; hence $Z_1 = r^{1/2}(0.5 - S)$ for r odd and $Z_1 = r^{1/2}[r/\{2(r+1)\} - S]$ for r even. Statistic S^* is $2\bar{z}$.

In order to calculate the statistics, values of m_i (or, more precisely, values of the difference $k_i = m_i - m_{i-1}$) are needed. For the normal distribution extensive tables can be found in Harter (1961), and are reproduced in *Biometrika Tables for Statisticians*, Vol. 2; also computer routines exist (for example in the IMS library of subroutines) to calculate the m_i very accurately. For the extreme-value distribution, tables of k_i are given for $3 \leqslant n \leqslant 25$ by Mann, Scheuer and Fertig (1973). For the logistic distribution $k_i = n/\{(i-1)(n-i+1)\}$, $i = 2, \ldots.,$ n.

The statistics A^2, Z_1 and Z_2 are functionals of the quantile process $Q_n(t)$ of the z_i, and of the empirical process $R_n(t)$, where

$$Q_n(t) = r^{1/2}(z_{(v)} - t), \quad 0 \leqslant t \leqslant 1,$$

(here v is the greatest integer in $(r+1)t$, and $z_{(0)} \equiv 0$ and $z_{(r+1)} \equiv 1$ by definition), and

$$R_n(t) = r^{1/2}\left[r^{-1} \sum_{1}^{r} I(z_{(i)} \leqslant t) - t \right] \qquad 0 \leqslant t \leqslant 1.$$

Here $I(B)$ is the indicator function; $I(B) = 1$ if event B occurs, and $I(B) = 0$ otherwise. Specifically, it may be shown that

$$Z_1 = Q_n(\tfrac{1}{2}) + o_p(1);$$

$$Z_2 = \int_0^1 Q_n(t)\, dt + o_p(1); \text{ and}$$

$$A^2 = \int_0^1 R_n^2(s) ds/\{s(1-s)\}.$$

Suppose distribution F has density f with derivative $\dot{f}$, and let $F^{-1}(\cdot)$ be the inverse of F, that is $F^{-1}(x) = \inf(t\colon F(t) \geqslant x\}$.

Define $c(x) = -(1 + (1-x)\dot{f}(F^{-1}(x))/f^2(F^{-1}(x)))$, and set

$$I_1(s) = \int_0^s (1 + uc(u))/(1-u) du, \quad I_2(s) = \int_0^s c(u) I_1(u) du, \qquad \text{and} \quad I_3(s, t) = \int_s^t c(x) dx.$$

$$\text{Set } \rho_0(t, s) = \rho_0(s, t) = s + 2I_2(s) + I_1(s) I_3(s, t), \quad \text{for} \quad 0 \leqslant s \leqslant t \leqslant 1. \tag{6}$$

Finally, with $0 \leqslant p < q \leqslant 1$, set $t^* = p + t(q-p)$ and $s^* = p + s(q-p)$ and let

$$\begin{aligned} \rho(t, s) = \rho(s, t) = (q-p)^{-1}\{\rho_0(t^*, s^*) - s\rho_0(t^*, q) - (1-s)\rho_0(t^*, p) \\ - t\rho_0(s^*, q) - (1-t)\rho_0(s^*, p) + st\rho_0(q, q) + (1-s)(1-t)\rho_0(p, p) \\ + (s + t - 2st)\rho_0(p, q)\}. \end{aligned} \tag{7}$$

To simplify notation we shall sometimes omit the arguments of, for example, $Q_n(t)$, and of $\rho(s, t)$, In order to develop the asymptotic theory of the test statistics A_1^2, Z_1 and Z_2, we require two results. The first is:

Result 1: When the location and scale parameters in $G(x)$ are $\alpha = 0$ and $\beta = 1$, the process $\eta_n(t) = n^{1/2}\left\{\left(\sum_{j=1}^{[nt]} y_j/n\right) - t\right\}$ converges weakly to a Gaussian process $\eta(t)$ with mean 0 and covariance ρ_0.

Let $k/n \to p$ and $(k + r + 1)/n \to q$ as $n \to \infty$. It follows from result 1 that $Q_n(t)$ above converges weakly to the Gaussian process $Q(t) = (q-p)^{-1/2}[\eta\{p+t(q-p)\} - t\eta(q) - (1-t)\eta(p)]$, and $R_n(t)$ then converges weakly to $R(t) = -Q(t)$.

Comment on Result 1: Csorgo (1983) has given a thorough discussion on empirical processes, a number of which are closely related to η_n. In fact, Csorgo and Revesz (1980) have proved that a process $\eta_n^*(t)$, very similar to $\eta_n(t)$, converges weakly to a Gaussian process $\eta^*(t)$ with the same mean 0 and covariance ρ_0 as $\eta(t)$ above. The process $\eta_n^*(t)$ is obtained from $\eta_n(t)$ by replacing $y_j/n = \{x_{(j)} - x_{(j-1)}\}/\{n(m_j - m_{j-1})\}$ by $\{x_{(j)} - x_{(j-1)}\}f\{F^{-1}([j-1]/[n+1])\}$. The convergence of $\eta_n(t)$ in Result 1 might then be proved by showing that $\eta_n^*(t)$ and $\eta_n(t)$ are sufficiently close. For the logistic distribution this is straightforward, but tedious; for other distributions, for example, the normal, extensive analysis is needed, which we have not attempted here.

For statistic A^2, we also require that, as $n \to \infty$, A^2 converges in distribution to $S = \int_0^1 [R^2(s)/\{s(1-s)\}]\, ds$. This should follow because $R_n(t)$ converges weakly to $R(t)$, but the result is difficult to prove because the denominator is zero at $s = 0$ and 1. This particular problem with A^2 has occurred in other goodness-of-fit situations; see, for example, Durbin (1973). However, it is precisely because the weighting of $R^2(s)$ becomes large towards the limits of the integral that A^2 is a good statistic—it gives weight to extreme observations. Hence it is worth giving results for A^2.

We note here that Result 1 and the convergence of A^2 both await rigorous proof. For this article we have verified by simulation studies that the A^2 distributions for finite n do converge to the appropriate asymptotic distribution. This distribution is that of $S = \Sigma\lambda_i\omega_i$, i $= 1, \ldots, \infty$, where ω_i are independent χ_1^2 variables, and where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$ are the eigenvalues of

$$\lambda f(s) = \int_0^1 f(t)\rho^*(s, t)dt, \tag{8}$$

with $\rho^*(s, t) = \rho(s, t)/\{s(1-s)t(1-t)\}^{1/2}$. This follows from well-known theory (see, for example, Durbin 1973).

The asymptotic distributions of Z_1 and Z_2 are respectively $N(0, \sigma_1^2)$ and $N(0, \sigma_2^2)$, where

$$\sigma_1^2 = \rho(\tfrac{1}{2}, \tfrac{1}{2}) \quad \text{and} \quad \sigma_2^2 = \int_0^1\int_0^1 \rho(s, t)\, ds\, dt. \tag{9}$$

In order to obtain asymptotic points and distributions, we therefore need $\rho(s, t)$ for various distributions. These are given in the next section.

3. COVARIANCE FUNCTIONS FOR SPECIFIC DISTRIBUTIONS

3.1. When $G(x)$ is the uniform distribution, it is easily shown that the $z_{(i)}$ are exactly distributed as the order statistics of a sample of size r from $U(0, 1)$. As was stated in Section 1, this is also true when $G(x)$ is the exponential distribution. The asymptotic distribution of $\eta_n(t)$ is then that of Brownian motion, and Q and R are Brownian bridges. The asymptotic distribution of A^2 is then the Case 0 distribution tabulated in Stephens (1974, 1976), and the asymptotic distributions of Z_1 and Z_2 are respectively $N(0,1/4)$ and $N(0, 1/12)$.

3.2. When $F(w) = \Phi(w)$, the standard normal distribution with density $\phi(w)$, we find

$$\begin{aligned}
c(x) &= (1-x)\Phi^{-1}(x)/\phi(\Phi^{-1}(x)),\\
I_1(s) &= [s + s(\Phi^{-1}(s))^2 + \Phi^{-1}(s)\phi(\Phi^{-1}(s))]/2,\\
I_2(s) &= -[(s^2+s)/4 + (s^2-s)(\Phi^{-1}(s))^2/2 + (2s-1)\Phi^{-1}(s)\phi(\Phi^{-1}(s))/4,\\
&\quad + (s^2-s)(\Phi^{-1}(s))^4/4 + (2s-1)(\Phi^{-1}(s))^3\phi(\Phi^{-1}(s))/4\\
&\quad + (\Phi^{-1}(s))^2\phi^2(\Phi^{-1}(s))/4]/2 \quad \text{and}\\
I_3(s, t) &= J(t) - J(s) \quad \text{for} \quad 0 < s \leqslant t < 1,
\end{aligned}$$

where

$$J(t) = [(\Phi^{-1}(t))^2(1 - t) - \Phi^{-1}(t)\phi(\Phi^{-1}(t)) - t]/2.$$

These are used in (6) and (7) to give $\rho(s, t)$.

3.3 For $F(x) = \exp\{-\exp(-x)\}$, the extreme value distribution, we find

$$c(x) = (\log x)^{-1} - x^{-1} - (x \log x)^{-1}, \quad I_1(s) = E_1(-\log s),$$

$$I_2(s) = -E_1^2(-\log s)/2 + (\log s - \log(-\log s))E_1(-\log s) - s + \int_{-\log s}^{\infty} y^{-1} \log(y) e^{-y} dy, \text{ and}$$

$$I_3(s, t) = K(s) - K(t) \ (0 < s \leqslant t < 1),$$

where

$$K(s) = E_1(-\log s) + \log(-\log s) + \log s \quad \text{and} \quad E_1(y) = \int_y^{\infty} x^{-1} e^{-x}\, dx.$$

These expressions are used in (6) and (7) to give $\rho(s, t)$. The extreme value distribution is sometimes written in the form $F^*(x) = 1 - \exp\{-\exp(x)\}$, $-\infty < x < \infty$. $F^*(x)$ is the distribution of $-x'$, where x' has the distribution $F(\cdot)$ at the beginning of this subsection. For $F^*(x)$, the covariance $\rho^*(s, t)$ is found from $\rho(s, t)$ for $F(x)$, by the relation $\rho^*(s, t) = \rho(1 - s, 1 - t)$.

3.4. For the logistic distribution, $F(x) = (1 + e^{-x})^{-1}$, we have

$$c(x) = (x - 1)/x,$$

$$I_1(s) = -s - \log(1 - s),$$

$$I_2(s) = s - s^2/2 + \int_0^s u^{-1}(1 - u) \log(1 - u) du, \quad \text{and}$$

$$I_3(s, t) = t - s + \log s - \log t, \quad 0 < s \leqslant t < 1;$$

$\rho(s, t)$ is again calculated from (6) and (7).

4. DISTRIBUTIONS AND PERCENTAGE POINTS

Statistic A^2. The eigenvalues of $\rho^*(s, t)$ in equation (8), were found by a discrete approximation of the integral. To high accuracy, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are then those of the matrix system

$$\lambda f\{(i - \tfrac{1}{2})/k\} = \left[\sum_{j=1}^{k} f\{(j - \tfrac{1}{2})/k\} \rho^*\{(i - \tfrac{1}{2})/k, (j - \tfrac{1}{2})/k\}\right]/k.$$

We found that $k = 100$ gave sufficient accuracy, in the sense that an increase in k did not significantly change the asymptotic percentage points of A^2, which are given by those of $T = \Sigma\, \lambda_i \omega_i$, i $= 1, \ldots, k$. These were evaluated by Imhof's method (Durbin and Knott, 1972). The percentage points are given in Table 1, for the normal and logistic distributions, for various $0 \leqslant p < q \leqslant 1$. (Recall that $p = k/n$ and $q = (k + r + 1)/n$, as $k, r, n \to \infty$). In the case of symmetric distributions such as these, censoring at p, q leads to the same points as censoring at $1 - q, 1 - p$ so that the tables are quite compact.

The asymptotic points change fairly slowly with the censoring pattern, and interpolation in the tables is straightforward and works well.

For finite samples, Monte Carlo points have been found for the normal test, and for uncensored samples of sizes $n = 20$ and for $n = 40$. For A^2, the points for finite n converge fairly quickly to the asymptotic points; we observed that use of the exact values for the m_i (rather

TABLE 1

Asymptotic percentage points for A^2, for samples from normal or logistic populations

Left censoring point, p	Right censoring point, q	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
Normal distribution								
0	1	0.955	1.066	1.211	1.422	1.798	2.191	2.728
0	0.75	1.056	1.183	1.350	1.592	2.026	2.479	3.100
0	0.50	1.098	1.232	1.409	1.667	2.129	2.612	3.273
0	0.25	1.133	1.273	1.459	1.730	2.215	2.722	3.416
0.25	0.75	1.178	1.324	1.518	1.800	2.306	2.835	3.559
0.25	0.50	1.225	1.381	1.587	1.889	2.430	2.996	3.770
Logistic distribution								
0	1	1.123	1.263	1.448	1.720	2.206	2.716	3.413
0	0.75	1.141	1.281	1.468	1.741	2.230	2.741	3.441
0	0.50	1.178	1.325	1.521	1.806	2.318	2.852	3.584
0	0.25	1.215	1.369	1.574	1.873	2.409	2.969	3.736
0.25	0.75	1.177	1.323	1.517	1.801	2.308	2.838	3.564
0.25	0.50	1.223	1.378	1.584	1.885	2.424	2.989	3.761

than, say Blom's approximation $m_i \simeq \Phi^{-1}\{(i-3/8)/(n+1/4)\}$ makes the convergence faster. Use of the asymptotic points at level α with finite samples gives a test level which is slightly greater than α.

Statistics Z_1 and Z_2. Statistics Z_1 and Z_2 are asymptotically normally distributed with mean 0, and variance given by (9). For the uncensored case, these have been worked out analytically. For the normal distribution $\sigma_1^2 = 3/16 = 0.1875$ and $\sigma_2^2 = (1 - 3^{1/2}/\pi)/8 = 0.056084$; for the logistic distribution the values are $\sigma_1^2 = 1 - \pi^2/12 + (0.5 - \log 2)^2 = 0.21484$, and $\sigma_2^2 = (\pi^2 - 9)/12 = 0.07247$. Statistics Z_1 and Z_2 converge quickly to their asymptotic distributions, as one would expect. Thus, to make a test for normality based on the median, we calculate $Z_1^* = \{r/0.1875\}^{1/2}(z_{(\{r+1\}/2} - 0.5)$ if r is odd, and refer Z_1^* to a standard normal distribution; if r is even, the bracket including z is replaced by $[z_{(\{r+1\}/2)} - (r+2)/\{2(r+1)\}]$. For the test based on the mean, $Z_2^* = \{r/0.0561\}^{1/2}(\bar{z} - 0.5)$ is referred to the standard normal distribution. Note that if the $z_{(i)}$ were ordered uniforms, the values of σ_1^2 and σ_2^2 would be 0.25 and 0.0833 respectively; the true asymptotic variances of Z_1 and Z_2 are much smaller, especially in the normal case.

Non-consistency and bias. Some calculations have also been made when the test is for the normal or the logistic distribution, but the sample tested is actually uniform. For the logistic test, statistics Z_1 and Z_2 are again asymptotically normal with mean 0; the variances $\sigma_1^2 = 0.3$ and $\sigma_2^2 = 3/70 = 0.04286$. The algebra involved in the calculations is extensive and is not included here. Straightforward calculations then show that the asymptotic power of Z_2, for a 5% test against a uniform alternative, is 0.011, that is statistic Z_2 is not consistent and also biased. For Z_1 the asymptotic power is 0.097, very low, showing that Z_1 is not consistent. Similar results hold for Z_1 and Z_2 in the test for normality against the uniform alternative; the asymptotic power of Z_1 is 0.11, so that Z_1 is not consistent, and that of Z_2 is 0.031, so that Z_2 is biased and inconsistent. The Monte Carlo studies in Section 6 below verify these results, which certainly weaken the appeal of these statistics.

5. EXAMPLE

Example. Table 2, part (a), gives 15 values of X, a measure of endurance of industrial specimens, taken from Section 6.2 of *Biometrika Tables for Statisticians*, Vol. 2. Graphical plots are given there and suggest that $x = \log X$ might be normally distributed. Also given in Table 2 are: the values $x_{(i)}$; values of m_i; the normalized spacings y_i; and the values $z_{(i)}$, together with the values of the test statistics. Reference to Table 1 shows that A^2 is not nearly significant, so that lognormality of the original values is acceptable. The values of Z_1^* and Z_2^* (Section 4 above) are -0.958 and -0.594 and these too are not significant.

In part (b) of Table 2 the calculations are shown for a censored sample consisting of the first 11 of the ordered X set; again normality can be accepted. If the original X are used without taking logarithms, values of A^2 are 7.424 for the whole set, and 3.262 for the censored set. Reference to Table 1, with $p = 0$ and $q = 1$ or 11/15, shows both of these to be significant at the 1% level. These results agree with results of other tests described in *Biometrika Tables for Statisticians.*

6. POWER COMPARISONS: TESTS OF NORMALITY

In this section, we examine the power of the tests for normality, for complete samples, taken from a set of alternative distributions. Table 3 gives the percentage of 5000 samples declared significant by the various test statistics. These tests are for sample sizes $n = 20$ and $n = 40$, and the test level is 5%. The test statistics compared are A^2 (NS), that is A^2 based on normalized spacings, Z_1 and Z_2, against the well-known Anderson-Darling statistic A^2 (Case 3) and the Shapiro-Wilk (1965) statistic W. In A^2 (Case 3), the Anderson-Darling statistic is calculated using values $z_{(i)} = G(x_{(i)})$, with estimates $\bar{x}$ and s^2 for the normal distribution parameters μ and σ^2. Critical points are given by Stephens (1974).

TABLE 2

Values X of endurance measurements and calculations for A^2, Z_1^* and Z_2^*

Part (a)					
Values $X_{(i)}$:	0.20	0.33	0.45	0.49	0.78
	0.92	0.95	0.97	1.04	1.71
	2.22	2.275	3.65	7.00	8.80
Values $x_{(i)}$:	−1.609	−1.109	−0.799	−0.713	−0.248
	−0.084	−0.051	−0.030	0.039	0.536
	0.798	0.822	1.295	1.946	2.175
m_i	−1.736	−1.248	−0.948	−0.715	−0.516
	−0.335	−0.165	0.000	0.165	...
y_i	1.026	1.033	0.366	2.334	0.915
	0.189	0.126	0.422	2.925	1.447
	0.123	2.031	2.169	0.469	
$z_{(i)}$	0.066	0.132	0.156	0.306	0.364
	0.376	0.385	0.412	0.598	0.692
	0.700	0.831	0.970		
$A^2 = 0.375$					
Median $z_{(7)} = 0.385$	$Z_1^* = (13/0.1875)^{1/2}(0.385 - 0.5) = -0.958$				
Mean $\bar{z} = 0.461$	$Z_2^* = (13/0.0561)^{1/2}(0.461 - 0.5) = -0.594$				
Part (b) $z_{(i)}$	0.095	0.191	0.225	0.441	0.526
	0.544	0.555	0.595	0.866	
$A^2 = 0{\cdot}6067$					
Median $z_{(5)} = 0.526$	$Z_1^* = (9/0.1875)^{1/2}(0.526 - 0.5) = 0.104$				
Mean $\bar{z} = 0.449$	$Z_2^* = (9/0.0561)^{1/2}(0.449 - 0.5) = -0.646$				

The power studies show A^2 (NS), A^2 (Case 3) and W to have much the same power overall. A^2 (NS) detects skew alternatives better, and W and A^2 (Case 3) are better against symmetric alternatives. Z_1 and Z_2 are poor in power against symmetric alternatives; the results for the uniform and logistic alternatives, for example, verify the asymptotic results of Section 4, that Z_1 and Z_2 can be not consistent or even biased. Z_1 was originally introduced in connection with tests for the two-parameter Weibull distribution against a special class of alternatives, and was suggested as a one-tailed test. Here we have a different tested distribution and a wider range of alternatives and Z_1 and Z_2 have both been used as two-tailed tests. In their examination of Z_1, Z_2 and A^2 (NS) in connection with the tests for the Weibull distribution, Lockhart, O'Reilly and Stephens (1984) also found that A^2 (NS) has good power.

7. FINAL REMARKS

Tests based on normalized spacings have the considerable appeal in that they follow the same procedure for all distributions, and can be used with right-or left-censored samples. They avoid estimating unknown parameters, although values of m_i are required. It appears that A^2 (NS) is competitive in terms of power with other methods of testing, although more extensive comparisons are needed.

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TABLE 3

Power comparisons: Tests of normality. Test Level = 5%. The table gives the percentage of 5000 Monte Carlo samples declared significant by the appropriate statistics

Alternative	A^2 (*Case 3*)	*Shapiro-Wilk*	A^2 (*NS*)	Z_1	Z_2
		$n = 20$			
χ^2 1 d.f.	98	99	99	96	98
χ^2 3 d.f.	59	63	69	51	68
χ^2 4 d.f.	50	54	58	41	58
χ^2 10 d.f.	24	24	26	20	29
Exponential	79	83	87	72	82
Log Normal	93	94	96	90	95
Uniform	20	22	14	10	4
Logistic	11	10	9	6	10
Laplace	30	25	20	10	20
Cauchy	90	88	84	50	67
t_2	53	51	47	24	40
t_3	34	34	28	8	25
t_4	26	26	21	7	21
		$n = 40$			
χ^2 1 d.f.	100	100	100	100	100
χ^2 3 d.f.	97	98	97	88	96
χ^2 4 d.f.	82	89	92	76	90
χ^2 10 d.f.	39	44	50	38	53
Exponential	99	99	100	96	99
Log Normal	100	100	100	99	100
Uniform	46	62	43	11	4
Logistic	14	12	12	6	13
Laplace	50	42	36	10	21
Cauchy	100	99	98	62	76
t_2	79	75	71	30	49
t_3	51	50	45	18	31
t_4	35	36	32	12	26

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