Week 3 & 4
Randomized Blocks, Latin Squares, and Related Designs

Joslin Goh

Simon Fraser University
Outline

3.1 Randomized Complete Block Design

3.2 Balanced Incomplete Block Design

3.3 Latin Square Design

3.4 Graeco-Latin Square Design

3.5 Analysis of Covariance
3.1 Randomized Complete Block Design

**Blocking** is to systematically eliminate its effect on treatment effects. Within each block, subjects are assumed to be homogeneous. The variations within blocks should be less than the variations between blocks.

- **Nuisance factor:** a design factor that may have an effect on the response but is not of primary interest.
  - Unknown and uncontrollable: randomization
  - Known and uncontrollable: analysis of covariance
  - Known and controllable: blocking
Examples

a. Four training processes for IQ Score, 16 Students, each of 4 students are from the same department.

b. Test 3 crop varieties on 5 fields.

c. An industrial engineer is conducting an experiment on eye focus time. He is interested in the effect of the distance of the object from the eye on the focus time. Four different distances are of interest. He has five subjects available for the experiment.
\[ y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}, \quad i = 1, \ldots, a, \quad j = 1, \ldots, b \]

\( \mu \): grand/overall mean

\( \tau_i \): \( i \)th treatment effect, \( \sum_{i=1}^{a} \tau_i = 0 \)

\( \beta_j \): \( j \)th block effect, \( \sum_{j=1}^{b} \beta_j = 0 \)

\( \epsilon_{ij} \): experimental error \( \sim iid N(0, \sigma^2) \)
Estimates

\[ y_{ij} = \hat{\mu} + \hat{\tau}_i + \hat{\beta}_j + r_{ij} \]
\[
\sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \bar{y}_{..})^2 = b \sum_{i=1}^{a} (\bar{y}_{.i} - \bar{y}_{..})^2 + a \sum_{j=1}^{b} (\bar{y}_{.j} - \bar{y}_{..})^2 \\
+ \sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \bar{y}_{.i} - \bar{y}_{.j} + \bar{y}_{..})^2
\]

\[
SS_{Total} = SS_{Treatment} + SS_{Blocks} + SS_{Error}
\]
ANOVA Table

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Square</th>
<th>D.O.F</th>
<th>Mean Square</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>$b \sum_{i=1}^{a} (\bar{y}_i - \bar{y}..)^2$</td>
<td>a-1</td>
<td>$MS_{Treatments}$</td>
<td>$\frac{MS_{Treatments}}{MS_{Error}}$</td>
</tr>
<tr>
<td>Blocks</td>
<td>$a \sum_{j=1}^{b} (\bar{y}.j - \bar{y}..)^2$</td>
<td>b-1</td>
<td>$MS_{Blocks}$</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>$SS_{Total} - SS_{Treatments} - SS_{Blocks}$</td>
<td>(a-1)(b-1)</td>
<td>$MS_{Error}$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>$\sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \bar{y}..)^2$</td>
<td>N-1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[
\begin{align*}
E(\text{MS}_{\text{Error}}) &= \sigma^2 \\
E(\text{MS}_{\text{Treatments}}) &= \sigma^2 + \frac{b \sum_{i=1}^{a} \tau_i^2}{a - 1} \\
E(\text{MS}_{\text{Blocks}}) &= \sigma^2 + \frac{a \sum_{j=1}^{b} \beta_j^2}{b - 1}
\end{align*}
\]

Effective blocking
Hypothesis testing

\[ H_0 : \tau_1 = \tau_2 = \cdots = \tau_a \]

\[ H_1 : \exists i \neq j \text{ such that } \tau_i \neq \tau_j \]

If \( F > F_{\alpha, a-1, (a-1)(b-1)} \), reject \( H_0 \) at level \( \alpha \).
Assumption checking

- Normality: normal probability plot
- Independence:
- Constant variance:
- Additivity between blocks and treatments (Tukey’s non-additivity test)
Multiple comparisons

a. If $H_0$ is rejected, multiple comparisons of $\tau_i$’s should be performed.

b. Tukey multiple comparison method declares “treatments $i$ and $j$ are statistical different” if

$$|t_{ij}| = \left| \frac{\bar{y}_i - \bar{y}_j}{\hat{\sigma} \sqrt{1/b + 1/b}} \right| > \frac{1}{\sqrt{2}} q_{\alpha,a,(a-1)(b-1)}$$

c. Simultaneous CI for $\tau_i - \tau_j$ are

$$\bar{y}_i - \bar{y}_j \pm q_{\alpha,a,(a-1)(b-1)} \frac{\hat{\sigma}}{\sqrt{b}}$$
Example of Penicillin

Comparison of four variants of a penicillin product process

It was known that an important raw material, corn steep liquor, was quite variable. Each blend of corn steep liquor is sufficient for four runs. The experiment was protected from extraneous unknown sources of bias by running the treatments in random order within each block.

<table>
<thead>
<tr>
<th>Block (blend of corn steep liquor)</th>
<th>Treatment</th>
<th>Block average</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A B C D</td>
<td></td>
</tr>
<tr>
<td>blend 1</td>
<td>89(1) 88(3) 97(2) 94(4)</td>
<td>92</td>
</tr>
<tr>
<td>blend 2</td>
<td>84(4) 77(2) 92(3) 79(1)</td>
<td>83</td>
</tr>
<tr>
<td>blend 3</td>
<td>81(2) 87(1) 87(4) 85(3)</td>
<td>85</td>
</tr>
<tr>
<td>blend 4</td>
<td>87(1) 92(3) 89(2) 84(4)</td>
<td>88</td>
</tr>
<tr>
<td>blend 5</td>
<td>79(3) 81(4) 80(1) 88(2)</td>
<td>82</td>
</tr>
<tr>
<td>Treatment Average</td>
<td>84 85 89 86</td>
<td>86=Grand Average</td>
</tr>
</tbody>
</table>
plot(yield ~ blend+treat, data=penicillin)
Interaction plots

interaction.plot(penicillin$treat, penicillin$blend, penicillin$yield)
interaction.plot(penicillin$blend, penicillin$treat, penicillin$yield)
Fit the model

```r
> g <- lm(yield ~ treat+blend, penicillin)
> summary(g)
Call: lm(formula = yield ~ treat + blend, data = penicillin)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  90.000     2.745  32.791   4.1e-13 ***
treatB       1.000     2.745   0.364    0.72194
treatC       5.000     2.745   1.822    0.09351 .
treatD       2.000     2.745   0.729    0.48018
blendBlend2 -9.000     3.069  -2.933    0.01254 *
blendBlend3 -7.000     3.069  -2.281    0.04159 *
blendBlend4 -4.000     3.069  -1.304    0.21686
blendBlend5 -10.000    3.069  -3.259    0.00684 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1
Residual standard error: 4.34 on 12 degrees of freedom
Multiple R-squared: 0.5964,   Adjusted R-squared: 0.361
F-statistic: 2.534 on 7 and 12 DF,  p-value: 0.07535
```
Normality assumption

Normal Q–Q Plot

Theoretical Quantiles

Sample Quantiles

-2  -1  0  1  2
-4  -2  0  2  4  6

Theoretical Quantiles
Independence assumption
Constant variance assumption
Constant variance assumption

> bartlett.test(yield ~ treat, penicillin)

Bartlett test of homogeneity of variances

data: yield by treat
Bartlett’s K-squared = 0.6901, df = 3, p-value = 0.8755

> bartlett.test(yield ~ blend, penicillin)

Bartlett test of homogeneity of variances

data: yield by blend
Bartlett’s K-squared = 2.3859, df = 4, p-value = 0.6652
Tukey’s Non-additivity Test

Use package alr3

> tukey.nonadd.test(g)
  t value Pr(>|t|)
0.3134771 0.7539182
> anova(g)

Analysis of Variance Table

Response: yield

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>treat</td>
<td>3</td>
<td>70.000</td>
<td>23.333</td>
<td>1.2389</td>
<td>0.33866</td>
</tr>
<tr>
<td>blend</td>
<td>4</td>
<td>264.000</td>
<td>66.000</td>
<td>3.5044</td>
<td>0.04075 *</td>
</tr>
<tr>
<td>Residuals</td>
<td>12</td>
<td>226.000</td>
<td>18.833</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

Signif. codes:  0 ***  0.001 **  0.01 *  0.05 .  0.1  1
Fitting the model with the interaction term

```r
> alpha <- c(0,g$coef[2:4])
> alpha

         treatB treatC treatD
    0     1     5     2
> beta <- c(0,g$coef[5:8])
> beta

          blendBlend2 blendBlend3 blendBlend4 blendBlend5
    0       -9       -7       -4       -10
> ab <- rep(alpha,5) * rep(beta,rep(4,5))
> h <- update(g,.˜.+ab)
> anova(h)
Analysis of Variance Table
Response: yield

    Df Sum Sq Mean Sq F value  Pr(>F)
  treat  3  70.00  23.333  1.1458 0.37360
  blend  4 264.00  66.000  3.2411 0.05488 .
   ab    1   2.00   2.000  0.0983 0.75978
Residuals 11 223.99 20.364
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1 1
```
Multiple comparisons

- Fisher’s LSD ("Least Significant Difference")
- Bonferroni method
- Tukey method
Example 2: This experiment is to assess the effects of four different cholesterol-reducing diets on persons who have hypercholesterolemia. To run a CRBD, the blocks are chosen to represent combinations of the various gender-age-body size categories of interest. For each block of subjects, the four diets are randomly assigned to the sample of four persons in the block. Each subject is then followed for one year, after which the change in cholesterol level is recorded.

- Gender: Male/Female
- Age: Above/Below 50
- Body Size: Quetelet index above/below 3.5
Factor: Diet
Treatments: Diet 1, 2, 3, 4
```r
> g <- lm(change ~ diet + block, cholesterol)
> summary(g)

Call: lm(formula = change ~ diet + block, data = cholesterol)

Coefficients: Estimate Std. Error t value Pr(>|t|)
  (Intercept) 11.7156 0.5680 20.625 2.03e-15 ***
  dietB    -2.2125 0.4844 -4.567 0.000167 ***
  dietC    -2.1125 0.4844 -4.361 0.000274 ***
  dietD    -2.6375 0.4844 -5.445 2.11e-05 ***
  blockb2   -4.9000 0.6851 -7.153 4.72e-07 ***
  blockb3    4.6250 0.6851  6.751 1.12e-06 ***
  blockb4   -3.0750 0.6851 -4.489 0.000202 ***
  blockb5    2.8250 0.6851  4.124 0.000483 ***
  blockb6   -1.1750 0.6851 -1.715 0.101038
  blockb7    7.5000 0.6851 10.948 3.88e-10 ***
  blockb8    1.0750 0.6851  1.569 0.131552

---

Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1

Residual standard error: 0.9688 on 21 degrees of freedom
Multiple R-squared: 0.9618,   Adjusted R-squared: 0.9436
F-statistic: 52.89 on 10 and 21 DF,  p-value: 1.312e-12
```
> anova(g)

Analysis of Variance Table

Response: change

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>diet</td>
<td>3</td>
<td>33.56</td>
<td>11.19</td>
<td>11.918</td>
<td>9.011e-05 ***</td>
</tr>
<tr>
<td>block</td>
<td>7</td>
<td>462.86</td>
<td>66.12</td>
<td>70.445</td>
<td>3.742e-13 ***</td>
</tr>
<tr>
<td>Residuals</td>
<td>21</td>
<td>19.71</td>
<td>0.94</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
"Gathering faculty teaching evaluations by in-class and online surveys: their effects on response rates and evaluations" by Curt J. Dommeyer, Paul Baum, Robert W. Hanna and Kenneth S. Chapman

Experimental Design: The study was conducted using undergraduate business majors at California State University, Northridge. A total of 16 instructors participated in the study. Although the sample of instructors represents a convenience sample, the courses represent a cross-section of lower and upper division core courses that are required for business majors.

<table>
<thead>
<tr>
<th>online treatment</th>
<th>Accounting</th>
<th>Business Law</th>
<th>Economics</th>
<th>Finance</th>
<th>Management</th>
<th>Marketing</th>
<th>Management Sci</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grade incentive</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>Grade feedback</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Incentive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Demo</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>1</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Control group</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>Total</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>16</td>
</tr>
</tbody>
</table>
Each of the instructors in this study was assigned to have one of his/her sections evaluated in-class and the other evaluated online. In the online evaluation, each instructor was assigned either to a control group or to one of the following online treatments:
1. a very modest grade incentive (one-quarter of a percent) for completing the online evaluations;
2. an in-class demonstration of how to log on to the web site and complete the form;
3. an early grade feedback incentive in which students were told they would receive early feedback of their course grades (by postcard and/or posting the grades online) if at least two-thirds of the class completed online evaluations.
Seven different hardwood concentrations are being studied to determine their effect on the strength of the paper produced. However, the pilot plant can only produce three runs each day. As days may differ, the analyst uses the following design.

<table>
<thead>
<tr>
<th>Hardwood Concentration(%)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>114</td>
<td>120</td>
<td>117</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>126</td>
<td>120</td>
<td>119</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>137</td>
<td>117</td>
<td>134</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>141</td>
<td>129</td>
<td>149</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>145</td>
<td>150</td>
<td>143</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>120</td>
<td>118</td>
<td>123</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>136</td>
<td>130</td>
<td>127</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A *Balanced Incomplete Block Design* $BIBD(N,a,k,b,r,\lambda)$ is an incomplete block design in which
\[ a: \text{ the number of treatments} \]
\[ k: \text{ the block size} \]
\[ b: \text{ the number of blocks} \]
\[ r: \text{ the number of times each treatment occurs} \]
\[ \lambda: \text{ the number of times each pair of treatments appears in the same block} \]
\[ N: \text{ the number of observations} \]
Some constraints

Note that for some values of $a, r, b, k$, there do not exist a BIBD. For example, $a = 8, r = 8, k = 4, b = 16$. 
## Examples

<table>
<thead>
<tr>
<th>block 1</th>
<th>A</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>block 2</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>block 3</td>
<td>B</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>block 4</td>
<td>A</td>
<td>B</td>
<td>D</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>block 1</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>block 2</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>block 3</td>
<td>B</td>
<td>C</td>
</tr>
</tbody>
</table>
Model

\[ y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}, \ i = 1, \ldots, a, \ j = 1, \ldots, b \]

\( \mu \): overall mean

\( \tau_i \): \( i \)th treatment effect, \( \sum_i \tau_i = 0 \)

\( \beta_j \): \( j \)th block effect, \( \sum_j \beta_j = 0 \)

\( \epsilon_{ij} \): random error \( \epsilon_{ij} \ \text{iid} \sim N(0, \sigma^2) \)

Note:

1. Not all \( y_{ij} \) exists

2. Treatments and blocks are not orthogonal (independent)
Estimation

\[ \hat{\mu} = \bar{y}_{..}; \quad \hat{\tau}_i = \frac{kQ_i}{\lambda a}; \quad \hat{\beta}_j = \frac{rQ_j'}{\lambda b} \]

where

- \( Q_i = y_{i.} - \frac{1}{k} \sum_{j=1}^{b} n_{ij} y_{j.} \); \( Q_j' = y_{.j} - \frac{1}{r} \sum_{i=1}^{a} n_{ij} y_{i.} \);
- \( n_{ij} \) is 1 if treatment \( i \) appears in block \( j \) and 0 otherwise;
- \( \sum_i \hat{\tau}_i = 0; \sum_j \hat{\beta}_j = 0; \sum_i Q_i = 0; \)
- \( \text{Var}(Q_i) = \frac{(k-1)r}{k} \sigma^2 \) and \( \text{Var}(\hat{\tau}_i) = \frac{k(a-1)}{\lambda a^2} \sigma^2; \)
- \( Q_i \) is the adjusted total for \( i \)th treatment and they are not independent \( (\text{Var}(\hat{\tau}_i - \hat{\tau}_j) = 2k\sigma^2/\lambda a) \)
Note the blocks are incomplete and thus $\bar{y}_i - \bar{y}..$ is not an unbiased estimate of $\tau_i$.

For example

$$E(\bar{y}_1.) = \mu + \tau_1 + (\beta_1 + \beta_2 + \beta_3)/3$$
$$E(\bar{y}..) = \mu$$
$$E(\bar{y}_1. - \bar{y}..) = \tau_1 + (\beta_1 + \beta_2 + \beta_3)/3$$
Define:

$$SS_{Total} = \sum_{i,j} (y_{ij} - \bar{y}..)^2 = \sum_{i,j} y_{ij}^2 - \frac{y^2..}{N}$$

$$SS_{Blocks} = k \sum_{j} (\bar{y}.j - \bar{y}..)^2 = \frac{1}{k} \sum_{j} y_{.j}^2 - \frac{y^2..}{N}$$

$$SS_{Treatments(adjusted)} = \frac{k \sum_{i=1}^{a} Q_i^2}{\lambda a}$$

$$SS_{Error} = SS_{Total} - SS_{Blocks} - SS_{Treatments(adjusted)}$$
<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>D.O.F.</th>
<th>Mean Squares</th>
<th>F $\frac{MS_{Treatments(adjusted)}}{SS_{Error}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks</td>
<td>SS$_{Blocks}$</td>
<td>$b - 1$</td>
<td>SS$_{Blocks}/(b - 1)$</td>
<td></td>
</tr>
<tr>
<td>Trts(adjusted)</td>
<td>SS$_{Treatments(adjusted)}$</td>
<td>$a - 1$</td>
<td>SS$_{Treatments(adjusted)}/(a - 1)$</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>SS$_{Error}$</td>
<td>$N - a - b + 1$</td>
<td>SS$_{Error}/(N - a - b + 1)$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>SS$_{Total}$</td>
<td>$N - 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Test statistic for testing the equality of the treatment effect is

$$F = \frac{MS_{Treatments(adjusted)}}{SS_{Error}}$$
Assessing Block Effects

To assess the block effects, we use

\[
SS_{Blocks(\text{adjusted})} = \frac{r \sum_{j=1}^{b} (Q'_j)^2}{\lambda b}
\]

and

\[
SS_{Total} = SS_{Treatments} + SS_{Blocks(\text{adjusted})} + SS_{Error}
\]

But

\[
SS_{Total} \neq SS_{Treatments(\text{adjusted})} + SS_{Blocks(\text{adjusted})} + SS_{Error}
\]
Confidence interval

For \( \tau_i - \tau_j \)

- Fisher LSD CI

\[
\hat{\tau}_i - \hat{\tau}_j \pm t_{\alpha/2,N-a-b+1} \sqrt{\frac{2k}{\lambda a} M S_{Error}}
\]

- Tukey’s CI

\[
\hat{\tau}_i - \hat{\tau}_j \pm \frac{q_{\alpha,a,N-a-b+1}}{\sqrt{2}} \sqrt{\frac{2k}{\lambda a} M S_{Error}}
\]
Suppose that a chemical engineer thinks that the time of reaction for a chemical process is a function of the type of catalyst employed. Four catalysts are currently being investigated. The experimental procedure consists of selecting a batch of raw material, loading the pilot plant, applying each catalyst in a separate run of the pilot plant, and observing the reaction time. Because variations in the batches of raw material may affect the performance of the catalysts, the engineer decides to use batches of raw material as blocks. However, each batch is only large enough to permit three catalysts to be run. Therefore a randomized incomplete block design must be used.

<table>
<thead>
<tr>
<th>Treatment (Catalyst)</th>
<th>Blocks (Batch of Raw Material)</th>
<th>( y_{i,j} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>73 74 - 71</td>
<td>218</td>
</tr>
<tr>
<td>2</td>
<td>- 75 67 72</td>
<td>214</td>
</tr>
<tr>
<td>3</td>
<td>73 75 68 -</td>
<td>216</td>
</tr>
<tr>
<td>4</td>
<td>75 - 72 75</td>
<td>222</td>
</tr>
</tbody>
</table>

\[ y_{..} = 221 224 207 218 \quad 870 = y_{..} \]
In the example, we have

\[ Q_1 = 218 - \frac{1}{3} (221 + 224 + 218) = -\frac{9}{3} \]

\[ Q_2 = 214 - \frac{1}{3} (207 + 224 + 218) = -\frac{7}{3} \]

\[ Q_3 = 216 - \frac{1}{3} (221 + 207 + 224) = -\frac{4}{3} \]

\[ Q_4 = 222 - \frac{1}{3} (221 + 207 + 218) = -\frac{20}{3} \]
For example:

$$SS_{Total} = \sum_{i,j} y_{ij}^2 - \frac{y^2}{N} = 63156 - \frac{870^2}{12} = 81$$

$$SS_{Blocks} = \frac{1}{k} \sum_j y_j^2 - \frac{y^2}{N} = \frac{1}{3} (221^2 + 207^2 + 224^2 + 218^2) - \frac{870^2}{12} = 55$$

$$SS_{Treatments(adjusted)} = \frac{[(-9/3)^2 + (-7/3)^2 + (-4/3)^2 + (20/3)^2]}{2*4} = 22.75$$

$$SS_{Error} = 81 - 22.75 - 55 = 3.25$$
Example

Incorrect analysis for testing equality of treatment effects

```r
> g<-lm(y ~ Treatment + Block, catalyst)
> anova(g)
```

Analysis of Variance Table

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>3</td>
<td>11.667</td>
<td>3.889</td>
<td>5.9829</td>
<td>0.0414634 *</td>
</tr>
<tr>
<td>Block</td>
<td>3</td>
<td>66.083</td>
<td>22.028</td>
<td>33.8889</td>
<td>0.0009528 ***</td>
</tr>
<tr>
<td>Residuals</td>
<td>5</td>
<td>3.250</td>
<td>0.650</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1
Correct analysis for testing equality of treatment effects

```
> g<-lm(y ~ Block + Treatment , catalyst)
> anova(g)

Analysis of Variance Table
Response: y

           Df Sum Sq Mean Sq  F value    Pr(>F)
Block       3 55.000  18.333  28.205 0.001468 **
Treatment   3 22.750   7.583  11.667 0.010739 *
Residuals   5  3.250   0.650
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1
```
#set the parameters
a<-4  b<-4  k<-3  r<-3  n<-12
lambda<-r*(k-1)/(a-1)

# create n_ij
Nij<-matrix(0,a,a)
for(i in 1:n) Nij[as.integer(Treatment[i]),as.integer(Block[i])]<-1

# compute y_i.
ysumi<-vector('numeric',a)
for(i in 1:a) ysumi[i]<-sum(y[((i-1)*r+1):(i*r)])

#compute y_.j
ysumj<-vector('numeric',b)
for(i in 1:n) {ysumj[as.integer(Block[i])]<- ysumj[as.integer(Block[i])] + y[i]}

# compute Q_i
Qi<-vector("numeric",a)
for(i in 1:a) Qi[i]<-ysumi[i]-sum(Nij[,]*ysumj)/k

#compute hat of tau_i
tauihat<-k*Qi/(lambda*a)
> # (1-\alpha)100\% Fisher LSD CI for tau_i - tau_j
> alpha<-0.05
> mse<-0.65
> for(i in 1:(a-1))
+ for(j in (i+1):a)
+ {
+ print(
+ c(tauhat[i]-tauhat[j]-qt(1-alpha/2,n-a-b+1)*sqrt(mse*2*k/(lambda*a)),
+ tauhat[i]-tauhat[j]+qt(1-alpha/2,n-a-b+1)*sqrt(mse*2*k/(lambda*a))))
+ }
[1]  -2.044811  1.544811
[1]  -2.419811  1.169811
[1]  -5.419811 -1.830189
[1]  -2.169811  1.419811
[1]  -5.169811 -1.580189
[1]  -4.794811 -1.205189
> #(1-alpha)100% Tukey CI for tau_i- tau_j
> alpha<-0.05
> mse<-0.65
> for(i in 1:(a-1))
+ for(j in (i+1):a)
+ {
+ print(
+ c(tauihat[i]-tauihat[j]-(qtukey(1-alpha,a,n-a-b+1)/sqrt(2))
+ *sqrt(mse*2*k/(lambda*a)),
+ tauihat[i]-tauihat[j]+(qtukey(1-alpha,a,n-a-b+1)/sqrt(2))
+ *sqrt(mse*2*k/(lambda*a))))
+ }
[1] -2.826341  2.326341
[1] -3.201341  1.951341
[1] -6.201341 -1.048659
[1] -2.951341  2.201341
[1] -5.9513415 -0.7986585
[1] -5.5763415 -0.4236585
Other incomplete block designs

- Youden square
- Partially balanced incomplete block design
- Cubic and rectangular Lattices
- Cyclic designs
Agronomy experiments: Suppose we want to test the relative effectiveness of 5 different fertilizer mixtures on oats. The five experiments cannot be carried out on the same plot of land. Even contiguous plots may vary in fertility because of a moisture gradient, different previous use of the land, or some other reasons. Dividing a single plot into a $5 \times 5$ grid of subplots, and administering the fertilizers according to a Latin square arrangement as in the figure below:
Typewriters experiments: A company is interested in purchasing one brand of typewriter from among the 5 kinds available in the market. To make a decision, a study is performed to evaluate the typewriter. It is suspected that the time of the day as well as the day of the week in which the test is carried out affects the output on the machine.

<table>
<thead>
<tr>
<th>Days</th>
<th>Shifts</th>
<th>Treatments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mon.</td>
<td>10 15 11 13 9</td>
<td>D B A C E</td>
</tr>
<tr>
<td>Tue.</td>
<td>10 12 16 10 12</td>
<td>E C B D A</td>
</tr>
<tr>
<td>Wed.</td>
<td>19 8 12 10 8</td>
<td>B E C A D</td>
</tr>
<tr>
<td>Thu.</td>
<td>9 12 10 14 12</td>
<td>A D E B C</td>
</tr>
<tr>
<td>Fri.</td>
<td>12 11 8 10 15</td>
<td>C A D E B</td>
</tr>
<tr>
<td>Total</td>
<td>60 58 57 57 56</td>
<td>288</td>
</tr>
</tbody>
</table>

The table above shows the number of minutes of output for each brand of typewriter on different days and shifts. The treatments are represented by letters D through E, indicating different brands.
Example: A hardness testing machine presses a pointed rod (the ‘tip’) into a metal specimen (a ‘coupon’), with a known force. The depth of the depression is a measure of the hardness of the specimen. It is feared that, depending on the kind of tip used, the machine might give different readings. The experimenter wants 4 observations on each of the 4 types of tips. Suppose that the ‘coupon’ and the ‘operator’ of the testing machine are thought to be factors. Suppose there are $p = 4$ operators, $p = 4$ coupons, and $p = 4$ tips. The first two are nuisance factors, the last is the treatment factor.

<table>
<thead>
<tr>
<th>Coupon</th>
<th>Operator</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1$</td>
<td></td>
<td>$A = 9.3$</td>
<td>$B = 9.3$</td>
<td>$C = 9.5$</td>
<td>$D = 10.2$</td>
</tr>
<tr>
<td>$i = 2$</td>
<td></td>
<td>$B = 9.4$</td>
<td>$A = 9.4$</td>
<td>$D = 10$</td>
<td>$C = 9.7$</td>
</tr>
<tr>
<td>$i = 3$</td>
<td></td>
<td>$C = 9.2$</td>
<td>$D = 9.6$</td>
<td>$A = 9.6$</td>
<td>$B = 9.9$</td>
</tr>
<tr>
<td>$i = 4$</td>
<td></td>
<td>$D = 9.7$</td>
<td>$C = 9.4$</td>
<td>$B = 9.8$</td>
<td>$A = 10$</td>
</tr>
</tbody>
</table>
Definition

Consider one treatment factor and two nuisance/blocking factors. A design is called Latin square design if each treatment appears exactly once in each row and in each column.

- The number of levels of each blocking factor must equal the number of levels of the treatment factor.

- Small run sizes

- Randomization
  - Randomly select a Latin square.
  - Randomize the experiment units and trial orders.
Model

\[ y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \epsilon_{ijk}, \quad i, j, k = 1, \ldots, p \]

\( y_{ijk} \): the response in row \( i \), column \( k \) using treatment \( j \)
\( \alpha_i \): row effect, \( \sum_i \alpha_i = 0 \)
\( \tau_j \): treatment effect, \( \sum_j \tau_j = 0 \)
\( \beta_k \): column effect, \( \sum_k \beta_k = 0 \)
\( \epsilon_{ijk}, \text{iid} \sim N(0, \sigma^2) \)
Estimation

- $\hat{\mu} = \bar{y}$...
- $\hat{\alpha}_i = \bar{y}_{i..} - \bar{y}$...
- $\hat{\tau}_j = \bar{y}_{.j} - \bar{y}$...
- $\hat{\beta}_k = \bar{y}_{..k} - \bar{y}$...
- $r_{ijk} = y_{ijk} - (\bar{y}_{i..} + \bar{y}_{.j} + \bar{y}_{..k}) + 2\bar{y}$...
ANOVA table

Define:

\[ SS_{Total} = \sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{...})^2 \]

\[ SS_{Rows} = p \sum_i (\bar{y}_{i..} - \bar{y}_{...})^2 \]

\[ SS_{Treatments} = p \sum_j (\bar{y}_{.j.} - \bar{y}_{...})^2 \]

\[ SS_{Columns} = p \sum_k (\bar{y}_{..k} - \bar{y}_{...})^2 \]

\[ SS_{Error} = \sum_i \sum_j \sum_k [y_{ijk} - (\bar{y}_{i..} + \bar{y}_{.j.} + \bar{y}_{..k}) + 2\bar{y}_{...}]^2 \]
<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>D.O. F.</th>
<th>Mean Squares</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>SS(_{Treatments})</td>
<td>(p - 1)</td>
<td>(SS_{Treatments}/(p - 1))</td>
<td>(\frac{MS_{Treatments}}{MS_{Error}})</td>
</tr>
<tr>
<td>Rows</td>
<td>SS(_{Rows})</td>
<td>(p - 1)</td>
<td>(SS_{Rows}/(p - 1))</td>
<td>(\frac{MS_{Rows}}{MS_{Error}})</td>
</tr>
<tr>
<td>Columns</td>
<td>SS(_{Columns})</td>
<td>(p - 1)</td>
<td>(SS_{Columns}/(p - 1))</td>
<td>(\frac{MS_{Columns}}{MS_{Error}})</td>
</tr>
<tr>
<td>Error</td>
<td>SS(_{Error})</td>
<td>((p - 2)(p - 1))</td>
<td>(SS_{Error}/[(p - 2)(p - 1)])</td>
<td>(\frac{MS_{Error}}{MS_{Error}})</td>
</tr>
<tr>
<td>Total</td>
<td>SS(_{Total})</td>
<td>(p^2 - 1)</td>
<td>(\frac{SS_{Total}}{(p^2 - 1)})</td>
<td>(\frac{MS_{Total}}{MS_{Error}})</td>
</tr>
</tbody>
</table>

Compare \(F\) to \(F_{\alpha,p-1,(p-2)(p-1)}\) to reject or accept \(H_0: \tau_1 = \cdots = \tau_p\).
> anova(g)

Analysis of Variance Table

Response: y

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>tip</td>
<td>3</td>
<td>0.38500</td>
<td>0.12833</td>
<td>38.5</td>
<td>0.0002585 ***</td>
</tr>
<tr>
<td>operator</td>
<td>3</td>
<td>0.82500</td>
<td>0.27500</td>
<td>82.5</td>
<td>2.875e-05 ***</td>
</tr>
<tr>
<td>coupon</td>
<td>3</td>
<td>0.06000</td>
<td>0.02000</td>
<td>6.0</td>
<td>0.0307958 *</td>
</tr>
<tr>
<td>Residuals</td>
<td>6</td>
<td>0.02000</td>
<td>0.00333</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1
Multiple comparisons

If $H_0$ is rejected, multiple comparisons of $\tau_i$ and $\tau_j$ should be performed. The test statistic is

$$t_{ij} = \frac{\bar{y}.i. - \bar{y}.j.}{\hat{\sigma} \sqrt{\frac{1}{p} + \frac{1}{p}}}$$

where $\hat{\sigma}^2$ is the mean square error in the ANOVA table.

Tukey multiple comparison method claims “treatments $i$ and $j$ are statistically significantly different” if

$$|t_{i,j}| > \frac{1}{\sqrt{2}} q_{\alpha,p,(p-2)(p-1)}$$
Crossover design is a type of randomized clinical trial. Each participant in the design is randomized to either group 1 or group 2. All participants in group 1 receive drug A in the first treatment period and drug B in the second period. All participants in group 2 receive drug B in the first treatment period and drug A in the second treatment period. Often there is a washout period between the two active drug periods. During the washout period, they receive no study medication. The purpose of the washout period is to reduce the likelihood that study medication taken in the first period will have an effect that carries over the next period.

<table>
<thead>
<tr>
<th>Latin Squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>Period 1</td>
</tr>
<tr>
<td>Period 2</td>
</tr>
</tbody>
</table>
Consider a clinical trial comparing Motrin versus placebo for the treatment of tennis elbow. Each participant was randomized to receive either Motrin (group 1) or placebo (group 2) for a 3-week period. All participants then had 2-week washout period during which they receive no study medication. All participants were then “crossed-over” for a second 3-week period to receive the opposite study medication from that initially received.

ANOVA table for crossover design

<table>
<thead>
<tr>
<th>Source</th>
<th>D.O.F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subjects (Columns)</td>
<td>9</td>
</tr>
<tr>
<td>Periods (Rows)</td>
<td>1</td>
</tr>
<tr>
<td>Treatments (Letters)</td>
<td>1</td>
</tr>
<tr>
<td>Error</td>
<td>18</td>
</tr>
<tr>
<td>Total</td>
<td>29</td>
</tr>
</tbody>
</table>
3.3 Graeco-Latin Squares

Definition: Orthogonal Latin squares
Two Latin squares are said to be orthogonal if each pair of letters appears exactly once in the superimposed squares. The superimposed square is called Graeco-Latin square.

\[
\begin{array}{ccc}
A\alpha & B\beta & C\gamma \\
B\gamma & C\alpha & A\beta \\
C\beta & A\gamma & B\alpha \\
\end{array}
\]
Model

\[ y_{ijkl} = \mu + \theta_i + \tau_j + \omega_k + \phi_l + \epsilon_{ijkl}, \quad i, j, k, l = 1, \ldots, p \]

- \( y_{ijkl} \): the observation in row \( i \) and column \( l \) for Latin letter \( j \) and Greek letter \( k \).
- \( \theta_i \): \( i \)th row effect
- \( \tau_j \): \( j \)th Latin letter treatment effect
- \( \omega_k \): \( k \)th Greek letter treatment effect
- \( \phi_l \): \( l \)th column effect
- \( \epsilon_{ijkl} \): iid \( \sim N(0, \sigma^2) \)
# ANOVA Table

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>D.O.F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rows</td>
<td>( p \sum_i (\bar{y}<em>{i\ldots} - \bar{y}</em>{\ldots})^2 )</td>
<td>( p - 1 )</td>
</tr>
<tr>
<td>Latin letter treatments</td>
<td>( p \sum_j (\bar{y}<em>{j\ldots} - \bar{y}</em>{\ldots})^2 )</td>
<td>( p - 1 )</td>
</tr>
<tr>
<td>Greek letter treatments</td>
<td>( p \sum_k (\bar{y}<em>{k\ldots} - \bar{y}</em>{\ldots})^2 )</td>
<td>( p - 1 )</td>
</tr>
<tr>
<td>Columns</td>
<td>( p \sum_l (\bar{y}<em>{l\ldots} - \bar{y}</em>{\ldots})^2 )</td>
<td>( p - 1 )</td>
</tr>
<tr>
<td>Error</td>
<td>( \text{SS}_{Error} ) (by substraction) ( (p - 3)(p - 1) )</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>( \sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \bar{y}_{\ldots})^2 )</td>
<td>( p^2 - 1 )</td>
</tr>
</tbody>
</table>

F-test and Tukey’s multiple comparison are similar as those in Latin square design.
### 3.5 Analysis of Covariance (ANCOVA)

Consider an experiment (Flurry, 1939) to study the breaking strength \( y \) in grams of three types of starch film. The breaking strength is also known to depend on the thickness of the film \( x \) as measured in \( 10^{-4} \) inches. Because film thickness varies from run to run and its values cannot be controlled or chosen prior to the experiment, it should be treated as a covariate whose effect on strength needs to be accounted for before comparing the three types of starch.

**Objective**: perform the treatment comparisons by incorporating the information of auxiliary covariate \( x \)
## Data for starch experiment

<table>
<thead>
<tr>
<th>Canna</th>
<th>Starch corn</th>
<th>Potato</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>791.7</td>
<td>7.7</td>
<td>731.0</td>
</tr>
<tr>
<td>610.0</td>
<td>6.3</td>
<td>710.0</td>
</tr>
<tr>
<td>710.0</td>
<td>8.6</td>
<td>604.7</td>
</tr>
<tr>
<td>940.7</td>
<td>11.8</td>
<td>508.8</td>
</tr>
<tr>
<td>862.7</td>
<td>11.7</td>
<td></td>
</tr>
<tr>
<td>592.5</td>
<td>7.2</td>
<td></td>
</tr>
</tbody>
</table>
Model

\[ y_{ij} = \mu + \tau_i + \gamma x_{ij} + \epsilon_{ij}, \quad i = 1, \ldots, k; \quad j = 1, \ldots, n_i \]

where

- \( \mu \): overall mean
- \( \tau_i \): \( i \)th treatment effect, \( \sum_{i=1}^{k} \tau_i = 0 \) or \( \tau_1 = 0 \)
- \( x_{ij} \): the covariate value for \( y_{ij} \)
- \( \gamma \): the regression coefficient for the covariate
- \( \epsilon_{ij} \): \( iid \sim N(0, \sigma^2) \)

Special cases
a. \( \gamma x_{ij} = 0 \), reduces to complete randomized design
b. \( \tau_i = 0 \), reduces to simple linear regression
Estimation

Assume $\tau_1 = 0$

We rewrite the model such that

$$y = X\beta + \epsilon$$

where

$$y = \begin{pmatrix} y_{11} \\ \vdots \\ y_{1n_1} \\ \vdots \\ y_{k1} \\ \vdots \\ y_{kn_k} \end{pmatrix}, \quad x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & x_{11} \\ 1 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & x_{1n_1} \\ 1 & 1 & 0 & 0 & 0 & x_{21} \\ 1 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 & 0 & x_{2n_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 1 & x_{k1} \\ 1 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 1 & x_{kn_k} \end{pmatrix}, \quad \beta = \begin{pmatrix} \mu \\ \tau_2 \\ \vdots \\ \tau_k \\ \gamma \end{pmatrix}, \quad \epsilon = \begin{pmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1n_1} \\ \vdots \\ \epsilon_{k1} \\ \vdots \end{pmatrix}$$
We have

$$\hat{\beta} = (X^T X)^{-1} X^T y = \begin{pmatrix} \hat{\mu} \\ \hat{\tau}_2 \\ \vdots \\ \hat{\tau}_k \\ \hat{\gamma} \end{pmatrix}$$

and thus we also obtain $\hat{\tau}_i - \hat{\tau}_j = \hat{\mu}_i - \hat{\mu}_j$
Hypothesis test

\[ H_0 : \tau_1 = \cdots = \tau_k \]

Model I: \[ y_{ij} = \mu + \gamma x_{ij} + \epsilon_{ij}, i = 1, \ldots, k; j = 1, \ldots, n_i \]

Model II: \[ y_{ij} = \mu + \tau_i + \gamma x_{ij} + \epsilon_{ij}, i = 1, \ldots, k; j = 1, \ldots, n_i \]

We have

\[
\text{SS(Reg, Model I)} = \text{SS}_{Total} - \text{SS}_{Error}(\text{Model I}) \\
\text{SS}_{Treatments} = \text{SS}_{Error}(\text{Model I}) - \text{SS}_{Error}(\text{Model II})
\]
<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Square</th>
<th>D.O.F.</th>
<th>Mean Square</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Covariate</td>
<td>SS(Reg, Model I)</td>
<td>1</td>
<td>SS(Reg, Model I)/1</td>
<td>MS_x/MS_{Error}</td>
</tr>
<tr>
<td>Treatments</td>
<td>SS_{Treatments}</td>
<td>k − 1</td>
<td>SS_{Treatments}/(k − 1)</td>
<td>MS_{Treatments}/MS_{Error}</td>
</tr>
<tr>
<td>Error</td>
<td>SS_{Error}(Model II)</td>
<td>\sum_i n_i − 1 − k</td>
<td>MS_{Error}</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>SS_{Total}</td>
<td>\sum_i n_i − 1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Multiple Comparisons

To compute $\text{Var}(\hat{\tau}_i - \hat{\tau}_j)$, note that

$$\text{Var}(\hat{\tau}_i - \hat{\tau}_j) = \text{Var}(\hat{\tau}_i) + \text{Var}(\hat{\tau}_j) - 2\text{Cov}(\hat{\tau}_i, \hat{\tau}_j).$$

Use the fact $\text{Var}(\hat{\beta}) = \sigma^2 (X^T X)^{-1}$ in multiple regression.

Test statistic $t_{ij} = \frac{\hat{\tau}_i - \hat{\tau}_j}{\sqrt{\text{Var}(\hat{\tau}_i - \hat{\tau}_j)}}$
> # set the model matrix
> x<-matrix(0,n,4)
> x[,1]<-rep(1,n)
> x[14:32,2]<-1
> x[33:49,3]<-1
> x[,4]<-dataset[,2]
> # fit the model
> g<-lm(y~x[,2]+x[,3]+x[,4])
> summary(g)

Call: lm(formula = y ~ x[, 2] + x[, 3] + x[, 4])

Coefficients: Estimate Std. Error t value Pr(>|t|)
(Intercept) 158.26 179.78 0.880 0.383360
x[, 2] -83.67 86.10 -0.972 0.336351
x[, 3] 70.36 67.78 1.038 0.304795
x[, 4] 62.50 17.06 3.664 0.000653 ***

---

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Residual standard error: 164.7 on 45 degrees of freedom
Multiple R-squared: 0.6815,    Adjusted R-squared: 0.6602
F-statistic: 32.09 on 3 and 45 DF,  p-value: 3.001e-11
<table>
<thead>
<tr>
<th>Effect</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>$t$ value</th>
<th>$p$ value</th>
</tr>
</thead>
<tbody>
<tr>
<td>intercept ($\mu$)</td>
<td>158.26</td>
<td>179.78</td>
<td>0.880</td>
<td>0.383360</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>62.50</td>
<td>17.06</td>
<td>3.664</td>
<td>0.000653</td>
</tr>
<tr>
<td>$\tau_2 - \tau_1$</td>
<td>-83.67</td>
<td>86.10</td>
<td>-0.972</td>
<td>0.336351</td>
</tr>
<tr>
<td>$\tau_3 - \tau_1$</td>
<td>70.36</td>
<td>67.78</td>
<td>1.038</td>
<td>0.304795</td>
</tr>
</tbody>
</table>
> starch<-vector("character",n)
> starch[1:13]<-"A"
> starch[14:32]<-"B"
> starch[33:49]<-"C"
> anova(lm(y~x[,4]+starch))   #ANOVA table

Analysis of Variance Table
Response: y

                     Df Sum Sq Mean Sq F value Pr(>F)
---
x[, 4]            1 2553357 2553357  94.1859 1.318e-12 ***
starch          2  56725   28362  1.0462 0.3597
Residuals        45 1219940   27110
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1
Incorrect analysis

```r
> anova(lm(y ~ starch + x[,4]))  # ANOVA table

Analysis of Variance Table
Response: y

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>starch</td>
<td>2</td>
<td>2246204</td>
<td>1123102</td>
<td>41.428</td>
<td>6.253e-11 ***</td>
</tr>
<tr>
<td>x[, 4]</td>
<td>1</td>
<td>363878</td>
<td>363878</td>
<td>13.422</td>
<td>0.0006526 ***</td>
</tr>
<tr>
<td>Residuals</td>
<td>45</td>
<td>1219940</td>
<td>27110</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1
> anova(lm(y ~ starch))  # ANOVA table ignoring thickness

Analysis of Variance Table

Response: y

Df  Sum Sq  Mean Sq   F value  Pr(>F)
  starch     2 2246204 1123102 32.619   1.512e-09 ***
Residuals  46 1583818   34431---

---

Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1
> solve(t(x) %*% x)

[1,]  1.1921580  -0.4774774  0.1170114  -0.1094193
[2,]  -0.4774774   0.2734201  0.0072684   0.0392997
[3,]   0.1170114  0.0072684  0.1694709  -0.0190275
[4,]  -0.1094193  0.0392997  -0.0190275   0.0107354

> txxinv<-solve(t(x) %*% x)


[1] 107.7622