

# Conjugate Prior Distribution for the Mean of a Gaussian

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## 1 The Set up

Consider the observed spread rate  $y$  and we wish to estimate the true average spread rate  $\mu$  given the data.

Likelihood:

$$y \sim N(\mu, \sigma^2)$$

prior on spread rate

$$\mu \sim N(\theta, \tau^2)$$

## 2 The Math work

Posterior

$$\begin{aligned} P(\mu \mid Y, \sigma^2, \tau^2, \theta) &\propto \frac{1}{\sqrt{\sigma^2 \tau^2}} \exp \left( \frac{-1}{2\sigma^2} \sum (y_i - \mu)^2 - \frac{1}{2\tau^2} (\mu - \theta)^2 \right) \\ &\propto \exp \left( \frac{-\sum (y_i^2 - 2y_i\mu + \mu^2)}{2\sigma^2} - \frac{(\mu^2 - 2\mu\theta + \theta^2)}{2\tau^2} \right) \\ &\propto \exp \left( \frac{-\tau^2 \sum (y_i^2 - 2y_i\mu + \mu^2) - \sigma^2 (\mu^2 - 2\mu\theta + \theta^2)}{2\sigma^2 \tau^2} \right) \\ &\propto \exp \left( -\frac{(-2\tau^2 n \bar{y} \mu + n\tau^2 \mu^2 + \sigma^2 \mu^2 - 2\sigma^2 \mu \theta)}{2\sigma^2 \tau^2} \right) \end{aligned}$$

$$\begin{aligned}
&\propto \exp\left(-\frac{\mu^2(n\tau^2 + \sigma^2) - 2\mu(\tau^2 n\bar{y} + \sigma^2\theta)}{2\sigma^2\tau^2}\right) \\
&\propto \exp\left(-\frac{(n\tau^2 + \sigma^2)}{2\sigma^2\tau^2} \left(\mu^2 - 2\mu\frac{(\tau^2 n\bar{y} + \sigma^2\theta)}{(n\tau^2 + \sigma^2)}\right)\right) \\
P(\mu \mid Y, \sigma^2, \tau^2, \theta) &= N\left(\frac{(\tau^2 n\bar{y} + \sigma^2\theta)}{(n\tau^2 + \sigma^2)}, \frac{\sigma^2\tau^2}{(n\tau^2 + \sigma^2)}\right)
\end{aligned}$$

The conclusion is pretty straight forward. Check it against intuition by setting  $\theta = 0$  and taking

$$\lim_{\tau \rightarrow \infty} P(\mu \mid Y, \sigma^2, \tau^2, \theta) = N(\mu, \sigma^2/n)$$

and you get back the usual frequentist.