On the discounted distribution functions of the surplus process perturbed by diffusion

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Abstract

In this paper, we derive explicit expressions for the discounted joint and marginal distribution functions of the surplus immediately prior to the time of ruin and the deficit at the time of ruin, and for the discounted distribution function of the amount of the claim causing ruin, based on the surplus process of ruin theory with an independent diffusion process. Furthermore, we show that these distribution functions satisfy defective renewal equations. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the classical continuous time risk model, the number of claims is assumed to follow a Poisson process \( \{ N(t) : t \geq 0 \} \) with parameter \( \lambda \). The individual claim sizes \( X_1, X_2, \ldots \), independent of \( N(t) \), are positive, independent and identically distributed random variables with common distribution function (df) \( P(x) = \Pr(X \leq x) \) and moments \( p_j = \int_0^\infty x^j \, dP(x) \) for \( j = 0, 1, 2, \ldots \). The aggregate process \( \{ S(t) : t \geq 0 \} \), where \( S(t) = X_1 + X_2 + \cdots + X_{N(t)} \) (with \( S(t) = 0 \) if \( N(t) = 0 \)) of the aggregate claims up to time \( t \), is a compound Poisson process with parameter \( \lambda \).

The surplus of the insurer at time \( t \) is

\[
U(t) = u + ct - S(t), \quad t \geq 0,
\]

where \( u = U(0) \) is the initial surplus, \( c = \lambda p_1(1 + \theta) \) is the constant rate per unit time at which the premiums are received, and \( \theta > 0 \) is the relative security loading.

Let \( T = \inf \{ t : U(t) < 0 \} \) be the time of ruin (the first time that the surplus becomes negative). Two important nonnegative random variables in connection with the time of ruin \( T \) are \( [U(T)] \), the deficit at the time of ruin, and \( U(T^-) \), the surplus immediately before the time of ruin, where \( U(T^-) \) is the left limit of \( U(T) \) at \( t = T \). Another associated random variable is \( [U(T^-) + |U(T)|] \), the amount of the claim causing ruin.

There have been many papers discussing the marginal and joint distributions of \( T, U(T^-) \) and \( |U(T)| \), based on the model (1.1) (see Gerber et al., 1987; Dufresne and Gerber, 1988; Dickson, 1992, 1993; Dickson and Waters,

In particular, Gerber and Shiu (1998) (see also Lin and Willmot, 1999) considered a function associated with a given penalty function \( w \) and the joint distributions of \( T, U(T^-) \) and \( |U(T)| \) as follows. For \( \delta \geq 0 \), define the expected discounted penalty function

\[
\phi_0(u) = E[e^{-\delta T} w(U(T^-), |U(T)|)I(T < \infty)|U(0) = u], \quad u \geq 0, \tag{1.2}
\]

where \( w(x, y), 0 \leq x, y < \infty \), is a nonnegative function; \( I \) is an indicator function such that \( I(T < \infty) = 1, T < \infty \) and \( I(T < \infty) = 0 \) otherwise. Of course, the ruin probability

\[
\psi_0(u) = E[I(T < \infty)|U(0) = u] = \Pr(T < \infty|U(0) = u), \quad u \geq 0 \tag{1.3}
\]

is the special case with \( w(x, y) = 1 \) and \( \delta = 0 \).

Dufresne and Gerber (1991) extended the classical risk model by adding an independent diffusion (or Wiener) process to (1.1) so that

\[
U(t) = u + ct - S(t) + \sigma W(t), \quad t \geq 0, \tag{1.4}
\]

where \( \sigma > 0 \) and \( \{W(t) : t \geq 0\} \) is a standard Wiener process that is independent of the compound Poisson process \( \{S(t) : t \geq 0\} \). In this case, there are two types of ruins: one is ruin caused by a claim \((T < \infty \text{ and } U(T) < 0)\), another is ruin due to oscillation \((T < \infty \text{ and } U(T) = 0)\). They also studied the probabilities of these two types of ruins, and proposed corresponding defective renewal equations, respectively.

Gerber and Landry (1998) generalized the discussion of Dufresne and Gerber (1991) based on the same model (1.4) by studying the expected discounted penalty function

\[
\phi(u) = w_0 \phi_d(u) + E[e^{-\delta T} w(U(T))I(T < \infty, U(T) < 0)|U(0) = u],
\]

and derived corresponding defective renewal equation, where \( w_0 \) is a nonnegative constant, \( w(-\gamma), y > 0 \), is a nonnegative function and \( \phi_d(u) = E[e^{-\delta T} I(T < \infty, U(T) = 0)|U(0) = u] \) is the Laplace transform or the expectation of the present value of the time of ruin \( T \) due to oscillation. Note that when \( \delta = 0 \), \( \phi_d(u) \) becomes the probability of ruin due to oscillation; if \( \delta > 0 \) and \( w(-\gamma) = 1 \), the second term on the right side of \( \phi(u) \) simplifies to the probability of ruin caused by a claim. Since \( w \) is a function of \( U(T) \) only, the applications of \( \phi(u) \) are restrictive in some aspects.

Tsai and Willmot (2001) further generalized the penalty function \( w \) by involving both \( U(T^-) \) and \( |U(T)| \). Then the corresponding expected discounted penalty function of interest is

\[
\phi_w(u) = E[e^{-\delta T} w(U(T^-), |U(T)|)|I(T < \infty, U(T) < 0)|U(0) = u]. \tag{1.5}
\]

With \( \phi_w(u) \) based on (1.4), many results like those which have been derived from (1.2) can be re-studied. When ruin occurs due to a claim, let \( f(x, y, t; D|u) \) (where \( D = \frac{1}{2} \sigma^2 \)) denote the defective joint probability density function of \( U(T^-), |U(T)| \) and \( T \). Define the discounted joint and marginal defective probability density functions of \( U(T^-) \) and \( |U(T)| \) based on model (1.4) with the discount factor \( \delta \geq 0 \) as follows:

\[
f(x, y; \delta, D|u) = \int_0^\infty e^{-\delta t} f(x, y, t; D|u) \, dt, \tag{1.6}
\]

\[
f_1(x; \delta, D|u) = \int_0^\infty f(x, y; \delta, D|u) \, dy = \int_0^\infty \int_0^\infty e^{-\delta t} f(x, y, t; D|u) \, dt \, dy, \tag{1.7}
\]

and

\[
f_2(y; \delta, D|u) = \int_0^\infty f(x, y; \delta, D|u) \, dx = \int_0^\infty \int_0^\infty e^{-\delta t} f(x, y, t; D|u) \, dt \, dx. \tag{1.8}
\]
Note that when \( D = 0 \), the surplus process (1.4) reduces to (1.1). In this case, the discounted probability density function is denoted by \( f(x, y; \delta, 0|u) \), \( f_1(x; \delta, 0|u) \), and \( f_2(y; \delta, 0|u) \), respectively. Dickson (1992) gave an explicit expression for \( f_1(x; 0, 0|u) \). Later, Gerber and Shiu (1998) generalized Dickson’s formula for the case \( \delta \geq 0 \) and got \( f_1(x; \delta, 0|u) \) by using a martingale approach. Lin and Willmot (1999), however, used some different methods to derive explicit expressions for \( f(x, y; 0, 0|u) \), \( f_1(x; 0, 0|u) \), and \( f_2(y; 0, 0|u) \).

Our goals are to derive explicit expressions for the discounted joint and marginal distribution and probability density functions of the surplus immediately prior to the time of ruin and the deficit at the time of ruin, and for the discounted distribution and probability density function of the amount of the claim causing ruin, based on the generalized surplus process (1.4). In addition, we will show that these distribution functions satisfy defective renewal equations.

Tsai and Willmot (2001) showed that \( \phi_w(u) \) in (1.5) satisfies the following defective renewal equation:

\[
\phi_w(u) = \frac{1}{1 + \beta} \int_0^u \phi_w(u - y) dG(y) + \frac{1}{1 + \beta} B(u),
\]

where

\[
B(u) = \frac{\lambda}{D} \int_0^u \int_e^{-b(u-s)} \int_0^e e^{-(x_1-s)} \int_0^{x_1} u(x_1, x_2 - x_1) dP(x_2) dx_1 ds,
\]

\( b = c/D + \rho \), \( \beta = \delta/(D\rho^2 + c\rho - \delta) \), and \( \rho = \rho(\delta, D) \) is the unique nonnegative root of generalized Lundberg’s equation \( \lambda \tilde{P}(\xi) = \lambda \int_0^\infty e^{-\xi x} dP(x) = \lambda + \delta - \beta \xi - D\xi^2 \) with \( \rho(0, D) = 0 \). In (1.9),

\[
G(y) = \int_0^y g(x) dx / \int_0^\infty g(x) dx = \int_0^y H(y - x) d\Gamma(x) = H \ast \Gamma(y)
\]
is called the associated ‘claim size’ distribution,

\[
g(y) = \frac{\lambda}{D} \int_0^y \int_e^{-b(y-s)} \int_0^e e^{-(x-\beta)} dP(x) ds
\]
is the discounted probability that the first record low is caused by a claim with

\[
\int_0^\infty g(y) dy = \frac{D\rho^2 + c\rho - \delta}{D\rho^2 + c\rho} = \frac{1}{1 + \beta} = \frac{\lambda}{b\lambda} \int_0^\infty e^{-\rho y} \tilde{P}(y) dy.
\]

\( H(x) = 1 - \tilde{H}(x) \) is a distribution function with \( \tilde{H}(x) = e^{-b/x} = H'(x)/b \), and \( \Gamma(x) = 1 - \tilde{\Gamma}(x) \) is also a distribution function with

\[
\tilde{\Gamma}(x) = \frac{\int_x^\infty e^{-\rho(y-x)} \tilde{P}(y) dy}{\int_0^\infty e^{-\rho y} \tilde{P}(y) dy}.
\]

Lin and Willmot (1999) demonstrated that any function \( \phi_w(u) \) satisfying (1.9) with arbitrary \( B(u) \) of bounded variation and distribution function \( G(u) \) may be expressed as

\[
\phi_w(u) = -\frac{1}{\beta} \int_0^u \tilde{K}(u - x) dB(x) + \frac{1}{\beta} B(u) - \frac{1}{\beta} B(0) \tilde{K}(u).
\]

Here \( \tilde{K}(u) = 1 - K(u) = \sum_{n=1}^\infty ((\beta/(1 + \beta))((1/(1 + \beta))^n \tilde{G}^{(n)}(u), u \geq 0 \), is a compound geometric distribution function with \( \tilde{K}(0) = 1/(1 + \beta) \), and \( K(u) \) satisfies the defective renewal equation

\[
\tilde{K}(u) = \frac{1}{1 + \beta} \int_0^u \tilde{K}(u - x) dG(x) + \frac{1}{1 + \beta} \tilde{G}(u), \quad u \geq 0.
\]
When $D = 0$, Tsai and Willmot (2001) showed that $\rho = \rho_0$, $G(u) = \bar{\Gamma}(u)$, $\beta = \beta_0$ and

$$\tilde{K}(u) = \tilde{K}_0(u) \equiv E[e^{-\delta T} I(T < \infty)|U(0) = u] = \sum_{n=1}^{\infty} \frac{\beta_0}{1 + \beta_0} \left( \frac{1}{1 + \beta_0} \right)^n \bar{\Gamma}^{*n}(u),$$

which satisfies the defective renewal equation

$$\tilde{K}_0(u) = \frac{1}{1 + \beta_0} \int_0^{u} \tilde{K}_0(u - x) \, d\Gamma_0(x) + \frac{1}{1 + \beta_0} \bar{\Gamma}_0(u), \quad u \geq 0 \tag{1.16}$$

with $\tilde{K}_0(0) = 1/(1 + \beta_0)$, where $\bar{\Gamma}_0(u) = \bar{\Gamma}(u)_{u=\rho_0}$ and $\rho_0 = \rho(\delta, 0)$ is the unique nonnegative root of Lundberg’s equation $\lambda \bar{\psi}(\xi) = \lambda \int_0^{\infty} e^{-\xi y} dP(x) = \lambda + \delta - c\xi$ with $\rho_0 = 0$ when $\delta = 0$.

2. Discounted joint distribution and probability density functions of $U(T-)$ and $|U(T)|$

Studying the discounted defective joint distribution function of $U(T-)$ and $|U(T)|$ helps one derive both the discounted defective marginal distribution functions of $U(T-)$ and $|U(T)|$. First of all, by appropriate choice of the penalty function $w(x, y)$, we have that the discounted defective joint distribution function of $U(T-)$ and $|U(T)|$ is equal to $\phi_w(u)$. Then an explicit expression for the discounted defective joint distribution function of $U(T-)$ and $|U(T)|$ can be obtained by (1.9) and (1.14). To see this, for any fixed $x$ and $y$, let

$$w(x_1, x_2) = \begin{cases} 1, & \text{if } x_1 \leq x, x_2 \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

By (1.6), $\phi_w(u)$ in (1.5) becomes $\phi_w(u) = \int_0^{x} \int_0^{y} \int_0^{\infty} e^{-\delta t} f(x_1, x_2, t; D|u) \, dt \, dx_1 \, dx_2 = \bar{F}(x, y; \delta, D|u)$, the discounted joint distribution function of $U(T-)$ and $|U(T)|$. The function $B(u)$ in (1.10) can be written using (1.12) and (1.13) as follows. If $0 \leq u < x$,

$$B(u) = \frac{\lambda}{D} (1 + \beta) \int_0^{u} e^{-b(u-s)} \int_s^{x} e^{-\rho(x_1-s)} \int_{x_1}^{x+y} p(x_2) \, dx_2 \, dx_1 \, ds$$

$$= \bar{G}(u) - \bar{G}(u + y) - \bar{H}(u)G(y) - \frac{b}{a} e^{-\rho x} (e^{OU} - e^{-Bu})[\bar{\Gamma}(x) - \bar{\Gamma}(x + y)], \tag{2.1}$$

and for $u > 0$

$$dB(u) = dG(u + y) - dG(u) + G(y) \, dH(u) - \frac{b}{a} e^{-\rho x} (\rho e^{OU} + b e^{-Bu})[\bar{\Gamma}(x) - \bar{\Gamma}(x + y)] \, du, \tag{2.2}$$

where $a = b + \rho = c/D + 2\rho$. If $0 < x \leq u$,

$$B(u) = \frac{\lambda}{D} (1 + \beta) \left[ \int_0^{x} + \int_x^{u} \right] e^{-b(u-s)} \int_s^{x} e^{-\rho(x_1-s)} \int_{x_1}^{x+y} p(x_2) \, dx_2 \, dx_1 \, ds$$

$$= \bar{H}(u - x)[\bar{G}(x) - \bar{G}(x + y)] - \bar{H}(u)G(y) - \frac{b}{a} e^{-\rho x} (e^{bs} - e^{-\rho x})[\bar{\Gamma}(x) - \bar{\Gamma}(x + y)], \tag{2.3}$$

and

$$dB(u) = -[\bar{G}(x) - \bar{G}(x + y)] \, dH(u - x) + G(y) \, dH(u) + \frac{b}{a} (e^{bs} - e^{-\rho x})[\bar{\Gamma}(x) - \bar{\Gamma}(x + y)] \, dH(u). \tag{2.4}$$

When $D = 0$, then $b/a = 1$, $\rho = \rho_0$, $G(x) = \Gamma_0(x)$ and $\bar{H}(u) = e^{-Bu} = 0$ for $u > 0$. In this case $B(u)$ becomes

$$B_0(u) = \begin{cases} \lfloor \bar{\Gamma}_0(u) - \bar{\Gamma}_0(u + y) \rfloor - e^{-\rho_0(x-u)}[\bar{\Gamma}_0(x) - \bar{\Gamma}_0(x + y)], & \text{if } 0 \leq u < x, \\ 0, & \text{if } 0 < x \leq u. \tag{2.5} \end{cases}$$
We remark that the expression above for $B_0(u)$, $0 \leq u < x$, also holds when $u = 0$, which cannot be obtained by letting $D = 0$ but by derivations similar to (2.1) from

$$B_0(u) = \frac{\lambda}{c}(1 + \beta_0) \int_u^\infty e^{-\rho_0(x_1-u)} \int_{x_1}^\infty w(x_1, x_2 - x_1) p(x_2) \, dx_2 \, dx_1. \quad (2.6)$$

Now we have the following result.

**Theorem 1.** For $D > 0$, the discounted defective joint distribution function of $U(T)$ and $|U(T)|$ is

$$F(x, y; D|u) = \begin{cases} 
\frac{1 + \beta}{\beta} [\tilde{K}(u) - \tilde{K}(u + y)] - \frac{1}{\beta} G(y) \tilde{K} * H(u) \\
+ \frac{1}{\beta} \int_0^y \tilde{K}(u + y - t) \, dG(t) + \frac{1}{\beta} e^{-\rho x} [\tilde{\Gamma}(x) - \tilde{\Gamma}(x + y)] \\
\times \left[ \rho \int_0^x e^{\rho t} \tilde{K}(u - t) \, dt + \tilde{\Gamma} * H(u) \right] \\
+ \frac{1}{\beta} \left[ \tilde{G}(x) - \tilde{G}(x + y) - \frac{b}{a} (\tilde{\Gamma}(x) - \tilde{\Gamma}(x + y)) \right] \tilde{K} * H(u - x), & \text{if } 0 \leq u < x,
\end{cases} \quad (2.7)$$

with $F(x, y; D|0) = 0$, where

$$\tilde{K} * H(u) = 1 - K * H(u) = \tilde{H}(u) + \tilde{K} * H(u). \quad (2.8)$$

For $D = 0$,

$$F(x, y; 0|u) = \begin{cases} 
\frac{1 + \beta_0}{\beta_0} [\tilde{K}_0(u) - \tilde{K}_0(u + y)] - \frac{1}{\beta_0} \Gamma(y) \tilde{K}_0(u) \\
+ \frac{1}{\beta_0} \int_0^y \tilde{K}_0(u + y - t) \, d\Gamma(t) + \frac{1}{\beta_0} e^{-\rho_0 x} [\tilde{\Gamma}(x) - \tilde{\Gamma}(x + y)] \\
\times \left[ \rho_0 \int_0^x e^{\rho_0 t} \tilde{K}_0(u - t) \, dt + \tilde{\Gamma}(u) \right] \\
+ \frac{1}{\beta_0} e^{-\rho x} [\tilde{\Gamma}(x) - \tilde{\Gamma}(x + y)] \rho_0 \int_0^x e^{\rho_0 t} \tilde{K}_0(u - t) \, dt + \tilde{\Gamma}(u), & \text{if } 0 \leq u < x,
\end{cases} \quad (2.9)$$

with $F(x, y; 0|0) = 1/(1 + \beta_0) [e^{-\rho_0 x} \Gamma(x) + \Gamma(y) - e^{-\rho_0 x} \Gamma(x + y)]$. 

For $D = 0$ and $\delta = 0$,

$$
F(x, y; 0, 0|u) = \begin{cases} 
\frac{1 + \theta}{\theta} \left[ \psi_0(u) - \psi_0(u + y) \right] + \frac{1}{\theta} \left[ P_1(x + y) - P_1(x) - P_1(y) \right] \psi_0(u) \\
- \frac{1}{\theta} \left[ P_1(x + y) - P_1(x) \right] + \frac{1}{\theta p_1} \int_0^x \psi_0(u + y - t) \tilde{P}(t) \, dt, & \text{if } 0 \leq u < x, \\
\frac{1}{\theta p_1} \int_0^x \psi_0(u - t) [\tilde{P}(t) - \tilde{P}(y + t)] \, dt \\
+ \frac{1}{\theta} \left[ P_1(x + y) - P_1(x) - P_1(y) \right] \psi_0(u), & \text{if } 0 < x \leq u
\end{cases}
$$

(2.10)

with $F(x, y; 0, 0|0) = (1/(1 + \theta)) [P_1(x) + P_1(y) - P_1(x + y)]$, where $P_1(x) = \int_0^x [1 - P(y)] \, dy/p_1$.

Proof. If $0 \leq u < x$, (1.14) turns out to be $F(x, y; \delta, D|u) = -(1/\beta) \int_0^{\beta u} \tilde{K}(u - t) \, dB(t) + (1/\beta) B(u)$ since $B(0) = 0$. Then replacing $B(u)$ and $B'(t)$ with (2.1) and (2.2), respectively, gives (2.7) by (1.15) for the case $0 \leq u < x$. For $u = 0$, $F(x, y; \delta, D|0) = ((1 + \beta)/\beta)[\tilde{K}(0) - \tilde{K}(y)] - (1/\beta) G(y) + (1/\beta) \int_0^\beta \tilde{K}(y - t) \, dB(t) = 0$ by (1.15) and $\tilde{K}(0) = 1/(1 + \beta)$.

Similarly, if $0 < x \leq u$, (1.14) becomes $F(x, y; \delta, D|u) = -(1/\beta) \int_0^\beta \tilde{K}(u - t) \, dB(t) - (1/\beta) \int_0^u \tilde{K}(u - t) \, dB(t) + (1/\beta) B(u)$. Substituting (2.3) and (2.4) for $B(u)$ and $B(t)$, respectively, leads to (2.7) for the case $0 < x \leq u$ after some algebra with the help of (1.15).

When $D = 0$, then $\beta = \beta_0$, $b/a = 1$, $\rho = \rho_0$, $G(y) = \Gamma_0(y)$, $\tilde{K}(u) = \tilde{K}_0(u)$, and for $u > 0, \tilde{K} * H(u) = \tilde{K}_0(u)$. Therefore, when $D = 0$, (2.7) simplifies to (2.9) for $u > 0$. For $u = 0$, the result can be obtained by derivations similar to (2.7) from $F(x, y; \delta, 0|u) = -(1/\beta_0) \int_0^{\beta_0 u} \tilde{K}_0(u - t) \, dB_0(t) + (1/\beta_0) B_0(u) - (1/\beta_0) B_0(0) \tilde{K}_0(u)$ with $B_0(u)$ given in (2.5). If $u = 0$, (2.9) becomes $F(x, y; \delta, 0|0) = (1/(1 + \beta_0)) [e^{-\beta_0 x} \Gamma(x) + \Gamma(y) - e^{-\beta_0 x} \Gamma(x + y)]]$ by (1.16) and $\tilde{K}_0(0) = 1/(1 + \beta_0)$.

If $\delta = 0$, then $\beta_0 = \theta$, $\rho = \rho_0$, $\tilde{K}_0(u)|_{\delta = 0} = \psi_0(u)$, $\Gamma_0(x) = P_1(x)$ and $d\Gamma(x) = \tilde{P}(x) \, dx/p_1$. In this case, (2.10) can be easily obtained from (2.9).

We remark that Tsai (2001) showed that $\tilde{K} * H(u)$ is equal to $\phi(u) = E[\psi_T I(T < \infty, U(T) = 0)|U(0) = u] + E[\psi_T T |(T < \infty, U(T) < 0)|U(0) = u]$, the sum of two expectations of the present value of the time of ruin $T$ caused by a claim and due to oscillation. For $\delta = 0$, $\phi_T(u)$ turns out to be $\psi_T(u) = \Pr(T < \infty, U(T) = 0|U(0) = u) + \Pr(T < \infty, U(T) < 0|U(0) = u)$, the probability of ruin.

Corollary 1. For $D > 0$, the discounted defective joint probability density function of $U(T -)$ and $|U(T)|$ is

$$
f(x, y; \delta, D|u) = \begin{cases} 
\frac{\lambda e^{-\rho x} p(x + y) e^{\rho x} - \tilde{K} * H(u) - \rho \int_0^u e^{\rho t} \tilde{K}(u - t) \, dt}{c + 2\rho D}, & \text{if } 0 \leq u < x, \\
\frac{\lambda e^{-\rho x} p(x + y) e^{\rho x} \tilde{K} * H(u - x) - \tilde{K} * H(u) - \rho \int_0^x e^{\rho t} \tilde{K}(u - t) \, dt}{c + 2\rho D}, & \text{if } 0 < x \leq u
\end{cases}
$$

(2.11)

with $f(x, y; \delta, D|0) = 0$. 

For $D = 0$,

$$f(x, y; 0|0) = \begin{cases} 
\frac{\lambda}{c} \frac{e^{-\rho x} p(x+y) e^{\rho (u-t)} \left( 1 - K_0(u) - \rho_0 \int_0^u e^{\rho t} \tilde{K}_0(u-t) \ dt \right)}{1 - K_0(0)}, & \text{if } 0 \leq u < x, \\
\frac{\lambda}{c} \frac{e^{-\rho x} p(x+y) e^{\rho (u-t)} \tilde{K}_0(u-x)}{1 - K_0(0)}, & \text{if } 0 < x \leq u 
\end{cases} \tag{2.12}$$

with $f(x, y; 0|0) = (\lambda/c) e^{-\rho x} p(x+y)$.

For $D = 0$ and $\delta = 0$,

$$f(x, y; 0|0) = \begin{cases} 
\frac{\lambda}{c} \frac{p(x+y) - \psi_0(u)}{1 - \psi_0(0)}, & \text{if } 0 \leq u < x, \\
\frac{\lambda}{c} \frac{p(x+y) \psi_0(u-x) - \psi_0(u)}{1 - \psi_0(0)}, & \text{if } 0 < x \leq u 
\end{cases} \tag{2.13}$$

with $f(x, y; 0|0) = (\lambda/c) p(x+y)$.

**Proof.** Since $f(x, y; \delta, D|u) = \partial^2 F(x, y; \delta, D|u)/\partial x \partial y$, if $0 \leq u < x$, (2.7) yields

$$\frac{\partial F(x, y; \delta, D|u)}{\partial y} = -\frac{1 + \beta}{\beta} \tilde{K}''(u + y) - \frac{1}{\beta} G'(y) \tilde{K} * H(u) + \frac{1}{\beta} G'(y) \tilde{K}(u) + \frac{1}{\beta} \int_y^\infty \tilde{K}''(u + y - t) \ dG(t) - \frac{1}{\beta} a e^{-\rho x} \tilde{K}'(x + y) \left[ \rho \int_0^u e^{\rho t} \tilde{K}(u-t) \ dt + \tilde{K} * H(u) - e^{\rho u} \right].$$

From (1.12) and (1.13),

$$\rho \tilde{F}(x) + \Gamma'(x) = -e^{\rho x} [e^{-\rho x} \tilde{F}'(x)]' = \tilde{P}(x)/\int_0^\infty e^{-\rho y} \tilde{P}(y) \ dy = \frac{\lambda}{bD} (1 + \beta) \tilde{P}(x). \tag{2.14}$$

Thus, $-(\partial/\partial x)[e^{-\rho x} \tilde{F}'(x+y)] = \rho e^{-\rho x} \tilde{F}'(x+y) + e^{-\rho x} \Gamma''(x+y) = e^{-\rho x} (\partial/\partial x)[\rho \tilde{F}(x+y) + \Gamma'(x+y)] = -e^{-\rho x} (\lambda/bD)(1 + \beta) p(x+y)$, which proves (2.11) for $0 \leq u < x$. If $0 < x \leq u$, (2.7) leads to

$$\frac{\partial F(x, y; \delta, D|u)}{\partial y} = -\frac{1}{\beta} \int_0^x \tilde{K}(u-t) G''(y+t) \ dt - \frac{1}{\beta} G'(y) \tilde{K} * H(u) + \frac{1}{\beta} \rho \tilde{K}(u-t) \ dt + \tilde{K} * H(u) - e^{\rho u} \left[ \tilde{G}'(x+y) - \frac{b}{a} \tilde{F}'(x+y) \right] \tilde{K} * H(u - x).$$

Since $K * H(u) = \int_0^u K(x) H'(u-x) \ dx = b \int_0^u e^{-b(x-u)} K(x) \ dx$, we have $[K * H(u)]' = b[K(u) - K * H(u)] = \lambda/bD - \beta K(u)$.
With Corollary 2. For $D > 0$, the discounted defective marginal distribution function of $|U(T)|$ is

$$
F_2(y; \delta, 0; u) = \frac{1 + \theta}{\theta} \int_0^y \check{K}(u + y - t) \, dG(t)
$$

with $F_2(y; \delta, 0; u) = 0$ and $F_2(\infty; \delta, \delta, D|u) = \lim_{y \to \infty} F_2(y; \delta, \delta, D|u) = ((1 + \beta)/\beta) \check{K}(u) - (1/\beta) \check{K}(u) \check{H}(u)$.

For $D = 0$,

$$
F_2(y; \delta, 0; u) = \frac{1 + \theta}{\theta} \int_0^y \check{K}(u + y - t) \, dG(t)
$$

with $F_2(y; \delta, 0; u) = 0$ and $F_2(\infty; \delta, 0; u) = \lim_{y \to \infty} F_2(y; \delta, 0; u) = \check{K}(u)$.

For $D = 0$ and $\delta = 0$,

$$
F_2(y; 0, 0; u) = \frac{1 + \theta}{\theta} \int_0^y \psi_0(u + y - t) \, d\check{P}(t)
$$

with $F_2(y; 0, 0; u) = (1/(1 + \theta)) \int_0^y \check{P}(t) \, dt$ and $F_2(\infty; 0, 0; u) = \lim_{y \to \infty} F_2(y; 0, 0; u) = \psi_0(u)$. 

3. Discounted distribution and probability density functions of $|U(T)|$

When ruin occurs due to a claim, one may be interested in the deficit at the time of ruin, $|U(T)|$, and its distribution function. Knowing the distribution function of $|U(T)|$ helps one evaluate the size of ruin caused by a claim. With the discounted joint distribution function of $U(T-)$ and $|U(T)|$, the discounted marginal distribution function of $|U(T)|$, $F_2(y; \delta, D|u)$, now is easily obtained from $F(x, y; \delta, D|u)$ by letting $x \to \infty$. 

From $\check{G}(x) = \check{G}(x) + G'(x)/b$ and $a = b + \rho$, the first term on the right side vanishes, and the second term turns out to be $-(1/\beta)(b/a)\rho \check{G}'(x) + \check{G}(x + y)\check{K}(u - x) = (1/\beta)(\rho/a) \cdot (1 + \beta)(\rho/\beta) \check{K} \check{H}(u - x) = (1/\beta)(\rho/\beta) \check{K} \check{H}(u - x)$ by (2.14). Combining this with the third term gives (2.11) for $0 < x < u$.

Similar arguments prove (2.12) using (2.9). If $\delta = 0$, then $\rho_0 = 0$, $\check{K}_0(u)|_{\delta = 0} = \psi_0(u)$, and (2.12) reduces to (2.13). □
Proof. By letting \( x \to \infty \) in the case \( 0 \leq u < x \) (since \( u \) is fixed) of (2.7), which implies that both \( \tilde{F}(x) \) and \( \tilde{F}(x+y) \to 0 \), one easily get (3.1). Eqs. (3.2) and (3.3) can be proved from (2.9) and (2.10), respectively, by similar arguments.

When \( y \to \infty \), \( \hat{K}(u + y) \to 0 \) and \( \int_0^y \hat{K}(u + y - t) \, dG(t) \leq \int_0^\infty \hat{K}(u + y - t) \, dG(t) \to 0 \), which imply that \( \lim_{y \to \infty} F_2(y; \delta, D|u) = ((1 + \beta)/\beta) \hat{K}(u) - (1/\beta) \hat{K} \ast \bar{H}(u) \).

\[ \Box \]

Note that Tsai (2001) showed that \( \lim_{y \to \infty} F_2(y; \delta, D|u) = ((1 + \beta)/\beta) \hat{K}(u) - (1/\beta) \hat{K} \ast \bar{H}(u) \) is equal to \( \phi_u(u) = E[e^{-\delta T} I(T < \infty, U(T) < 0)|U(0) = u] \), the expectation of the present value of the time of ruin \( T \) caused by a claim.

**Corollary 3.** For \( D > 0 \), the discounted defective probability density function of \( |U(T)| \) is

\[
 f_2(y; \delta, D|u) = -\frac{1 + \beta}{\beta} \hat{K}'(u + y) - \frac{1}{\beta} \bar{G}'(y) \bar{H} \ast \hat{K}(u) + \frac{1}{\beta} \int_0^y \hat{K}'(u + y - t) \, dG(t)
\]

(3.4)

with \( f_2(y; \delta, D|0) = 0 \).

For \( D = 0 \),

\[
 f_2(y; \delta, 0|u) = -\frac{1 + \beta_0}{\beta_0} \hat{K}'_0(u + y) + \frac{1}{\beta_0} \int_0^y \hat{K}'_0(u + y - t) \, dG_0(t)
\]

(3.5)

with \( f_2(y; \delta, 0|0) = (1/(1 + \beta_0)) \bar{G}'(y) \).

For \( D = 0 \) and \( \delta = 0 \),

\[
 f_2(y; 0, 0|u) = -\frac{1 + \theta}{\theta} \psi'(u + y) + \frac{1}{\theta \rho_1} \int_0^y \psi'(u + y - t) \hat{P}(t) \, dt
\]

(3.6)

with \( f_2(y; 0, 0|0) = (1/(1 + \theta)) \bar{P}'(y) = (\lambda/c) \hat{P}(y) \).

**Proof.** Differentiating (3.1) with respect to \( y \) gives

\[
 f_2(y; \delta, D|u) = -((1 + \beta)/\beta) \hat{K}'(u + y) - (1/\beta) \bar{G}'(y) [\bar{K} \ast \bar{H}(u) \tilde{F}(u)] + (1/\beta) \int_0^y \hat{K}'(u + y - t) \, dG(t).
\]

Combining this with \( \bar{K} \ast \bar{H}(u) - \tilde{K}(u) = \bar{H} \ast \bar{K}(u) \) yields (3.4). Similarly, differentiating (3.2) and (3.3) gives (3.5) and (3.6), respectively.

\[ \Box \]

**Theorem 2.** For \( D > 0 \), the discounted distribution function of \( |U(T)| \) satisfies the defective renewal equation

\[
 F_2(y; \delta, D|u) = \frac{1}{1 + \beta} \int_0^u F_2(y; \delta, D|u - x) \, dG(x) + \frac{1}{1 + \beta} [\tilde{G}(u) - \hat{G}(u + y) - G(y) \bar{H}(u)].
\]

(3.7)

For \( D = 0 \),

\[
 F_2(y; \delta, 0|u) = \frac{1}{1 + \beta_0} \int_0^u F_2(y; \delta, 0|u - x) \, dG_0(x) + \frac{1}{1 + \beta_0} [\bar{G}_0(u) - \hat{G}_0(u + y)].
\]

(3.8)

For \( D = 0 \) and \( \delta = 0 \),

\[
 F_2(y; 0, 0|u) = \frac{1}{1 + \theta} \int_0^u F_2(y; 0, 0|u - x) \, dP_1(x) + \frac{1}{1 + \theta} [\hat{P}_1(u) - \bar{P}_1(u + y)].
\]

(3.9)

which is (5) of Gerber et al. (1987).

**Proof.** We would like to prove (3.7) from a probabilistic viewpoint. Since \( g(x) \) defined in (1.11) is the discounted probability that the first record low (the first time where the surplus falls below the initial level) is caused by a jump where \( x \) is the amount by which the resulting first record low caused by a claim is below the initial surplus \( u \), and
\( H(u) = e^{-bu} \) is the expected discounted value of a contingent payment of 1 that is due at ruin, provided that ruin occurs before the first record low that is caused by a jump, by conditioning on the time and amount of the first record low caused by a claim, we obtain

\[
F_2(y; \delta, D|u) = \int_{0}^{u} F_2(y; \delta, D|u - x) g(x) \, dx + \int_{u}^{u+y} g(x) \, dx - H(u) \int_{0}^{y} g(x) \, dx. \tag{3.10}
\]

Note that the second term on the right side includes an unwanted contribution for the situation where ruin occurs by oscillation prior to the first record low caused by a jump; the third term is the corresponding offset.

Since \( \int_{0}^{\infty} g(x) \, dx = 1/(1 + \beta) \) and \( G'(x) = g(x)/\int_{0}^{\infty} g(z) \, dz \), (3.10) can be written as

\[
F_2(y; \delta, D|u) = (1/(1 + \beta)) \int_{0}^{u} F_2(y; \delta, D|u - x) \, dG(x) + (1/(1 + \beta)) \int_{u}^{u+y} \, dG(x) - (1/(1 + \beta)) H(u) \int_{0}^{y} \, dG(x),
\]

which leads to (3.7).

When \( D = 0 \), then \( \beta = \beta_0, G(x) = G_0(x) \), and \( H(u) = 0 \) for \( u > 0 \), which implies that (3.7) reduces to (3.8) for \( u > 0 \). If \( \delta = 0 \), then \( \beta_0 = \theta \) and \( G_0(x) = P_1(x) \), implying (3.9) from (3.8).

We remark that (3.7) can be shown by an algebraic approach as well. Please refer to Tsai (1999) for more details.

**Corollary 4.** For \( D > 0 \), the discounted defective probability density function of \( |U(T)| \) satisfies the defective renewal equation

\[
f_2(y; \delta, D|u) = \frac{1}{1 + \beta} \int_{0}^{u} f_2(y; \delta, D|u - x) \, dG(x) + \frac{1}{1 + \beta} [G'(u + y) - G'(y)H(u)]. \tag{3.11}
\]

For \( D = 0 \),

\[
f_2(y; \delta, 0|u) = \frac{1}{1 + \beta_0} \int_{0}^{u} f_2(y; \delta, 0|u - x) \, dG_0(x) + \frac{1}{1 + \beta_0} G'(u + y). \tag{3.12}
\]

For \( D = 0 \) and \( \delta = 0 \),

\[
f_2(y; 0, 0|u) = \frac{1}{1 + \theta} \int_{0}^{u} f_2(y; 0, 0|u - x) \, dP_1(x) + \frac{1}{1 + \theta} \bar{P}(u + y)/p_1. \tag{3.13}
\]

**Proof.** Differentiating (3.7)–(3.9) with respect to \( y \) yields (3.11)–(3.13), respectively. \( \square \)

Since \( F_2(\infty; \delta, D|u) = \lim_{y \to \infty} F_2(y; \delta, D|u) = \phi_0(u) \), \( F_2(y; \delta, D|u) \) is a discounted ‘defective’ distribution function. It is convenient to define the discounted ‘proper’ distribution function as \( F_{2,u}(y; \delta, D) = 1 - F_{2,u}(y; \delta, D)/\phi_0(u) \). Then we have the following result for \( F_{2,\infty}(y; \delta, D) = \lim_{u \to \infty} F_{2,u}(y; \delta, D) \).

**Theorem 3.** \( F_{2,\infty}(y; \delta, D) = \lim_{u \to \infty} F_{2,u}(y; \delta, D) \) satisfies

\[
F_{2,\infty}(y; \delta, D) = \frac{\int_{0}^{\infty} e^{\kappa x}[\tilde{G}(x) - \tilde{G}(x + y) - \tilde{G}(y)H(x)] \, dx}{\int_{0}^{\infty} e^{\kappa x}[\tilde{G}(x) - H(x)] \, dx}, \quad y \geq 0,
\]

where \( \kappa \) is positive and satisfies \( \int_{0}^{\infty} e^{\kappa x} \, dG(x) = 1 + \beta \).

**Proof.** Tsai (2001) showed that \( \phi_0(u) \) satisfies

\[
\phi_0(u) = \frac{1}{1 + \beta} \int_{0}^{u} \phi_0(u - x) \, dG(x) + \frac{1}{1 + \beta} \bar{G}(u),
\]

and from (3.7) \( F_2(y; \delta, D|u) \) satisfies

\[
F_2(y; \delta, D|u) = \frac{1}{1 + \beta} \int_{0}^{u} F_2(y; \delta, D|u - x) \, dG(x) + \frac{1}{1 + \beta} [G(u) - \tilde{G}(u + y) - \tilde{G}(y) \tilde{H}(u)].
\]
By the renewal theorem given by Feller (1971),

\[ \phi_s(u) = \frac{\int_0^\infty e^{sx} \int_0^\infty e^{sx} dG(x)}{\int_0^\infty e^{sx} dx} = \frac{\int_0^\infty e^{sx} \int_0^\infty e^{sx} dG(x)}{\int_0^\infty e^{sx} dx} = \frac{\int_0^\infty e^{sx} \int_0^\infty e^{sx} dG(x)}{\int_0^\infty e^{sx} dx} \to \infty, \]

and

\[ F_2(y; \delta, D|u) = \frac{\int_0^\infty e^{sx} [\tilde{G}(x) - \bar{G}(x + y) - G(y) \bar{H}(x)] dx}{\int_0^\infty e^{sx} dG(x)} \to \infty, \]

where \( \kappa \) is positive and satisfies \( \int_0^\infty e^{sx} dG(x) = 1 + \beta \).

Thus, \( F_2, \infty(y; \delta, D) = \lim_{u \to \infty} e^{sx} F_2(y; \delta, D|u) \phi_s(u) = \int_0^\infty e^{sx} [\tilde{G}(x) - \bar{G}(x + y) - G(y) \bar{H}(x)] dx / \int_0^\infty e^{sx} [\tilde{G}(x) - \bar{H}(x)] dx \), completing the proof.

We remark that since \( \int_0^\infty e^{sx} \tilde{G}(x + y) dx = e^{-\kappa y} \int_0^\infty e^{sx} \tilde{G}(t) dt, \int_0^\infty e^{sx} \bar{H}(x) dx = 1/(b - \kappa) \) by \( b > \kappa \), and \( \int_0^\infty e^{sx} \tilde{G}(x) dx = \beta/\kappa \), we also have

\[ F_2, \infty(y; \delta, D) = \frac{(\beta/\kappa) - (1/(b - \kappa))\tilde{G}(y) - e^{-\kappa y} \int_0^\infty e^{sx} \tilde{G}(t) dt}{(\beta/\kappa) - (1/(b - \kappa))}. \]

4. Discounted distribution and probability density functions of \( U(T^-) \)

When ruin occurs due to a claim, another important and interesting quantity is the surplus immediately before the time of ruin, \( U(T^-) \). With the discounted joint distribution function of \( U(T^-) \) and \( |U(T)| \), the discounted distribution function of \( U(T^-), F_1(x; \delta, D|u) \), is directly obtained from \( F(x, y; \delta, D|u) \) by letting \( y \to \infty \). Furthermore, there is a relationship between \( F_1(x; \delta, D|u) \) and \( F_2(y; \delta, D|u) \) shown in the section.

Corollary 5. For \( D > 0 \), the discounted defective marginal distribution function of \( U(T^-) \) is

\[ F_1(x; \delta, D|u) = \begin{cases} \frac{1 + \beta}{\beta} \bar{K}(u) - \frac{1}{\beta} \bar{K} \ast \bar{H}(u) + \frac{b}{\beta} e^{-\rho x} \tilde{G}(x) \\ \times \left[ \rho \int_0^u e^{ot} \bar{K}(u - t) dt + \bar{K} \ast \bar{H}(u) - e^{ou} \right], & \text{if } 0 \leq u < x, \\ \frac{1}{\beta} \int_0^x \bar{K}(u - t)G'(t) dt - \frac{1}{\beta} \bar{K} \ast \bar{H}(u) + \frac{b}{\beta} e^{-\rho x} \tilde{G}(x) \left[ \rho \int_0^x e^{ot} \bar{K}(u - t) dt + \bar{K} \ast \bar{H}(u) \right] \\ + \frac{1}{\beta} \tilde{G}(x) - \frac{b}{\beta} \tilde{G}(x) \bar{K} \ast \bar{H}(u - x), & \text{if } 0 < x \leq u \end{cases} \]

with \( F_1(x; \delta, D|0) = 0 \) and \( F_1(\infty; \delta, D|u) = \lim_{x \to \infty} F_1(x; \delta, D|u) = ((1 + \beta)/\beta)\bar{K}(u) - (1/\beta) \bar{K} \ast \bar{H}(u) \).
For $D = 0$,

$$F_1(x; \delta, 0|u) = \begin{cases} \tilde{K}_0(u) + \frac{1}{\beta_0} e^{-\rho_0 x} \tilde{G}_0(x) \left[ \rho_0 \int_t^u e^{\rho_0 t} \tilde{K}_0(u-t) \, dt + \tilde{K}_0(u) - e^{\rho_0 u} \right], & \text{if } 0 \leq u < x, \\
\frac{1}{\beta_0} \int_0^x \tilde{K}_0(u-t) \tilde{G}_0(t) \, dt - \frac{1}{\beta_0} \tilde{K}_0(u) \\
+ \frac{1}{\beta_0} e^{-\rho_0 x} \tilde{G}_0(x) \left[ \rho_0 \int_x^u e^{\rho_0 t} \tilde{K}_0(u-t) \, dt + \tilde{K}_0(u) \right], & \text{if } 0 < x \leq u \end{cases} \tag{4.2}$$

with $F_1(x; \delta, 0|0) = (1/(1 + \beta_0))[1 - e^{-\rho_0 x} \tilde{G}_0(x)]$ and $F_1(\infty; \delta, 0|u) = \lim_{x \to \infty} F_1(x; \delta, 0|u) = \tilde{K}_0(u)$.

For $D = 0$ and $\delta = 0$,

$$F_1(x; 0, 0|u) = \begin{cases} \left[ \frac{1}{\theta} \int_0^x \psi(u-t) \tilde{P}(t) \, dt - \frac{1}{\theta} \tilde{P}_1(u) \psi(u), \right. & \text{if } 0 \leq u < x, \\
\frac{1}{\theta} \int_0^x \psi(u-t) \tilde{P}(t) \, dt - \frac{1}{\theta} \tilde{P}_1(u) \psi(u), & \text{if } 0 < x \leq u \end{cases} \tag{4.3}$$

with $F_1(x; 0, 0|0) = (1/(1 + \theta)) \tilde{P}_1(x)$ and $F_1(\infty; 0, 0|u) = \lim_{x \to \infty} F_1(x; 0, 0|u) = \psi(u)$.

**Proof.** By letting $y \to \infty$ in (2.7), which implies that $\tilde{K}(u+y), \tilde{G}(x+y), \tilde{G}(y) \to 1$, and $\tilde{G}(0) \to 0, G(\infty) \to 1$, and $\int_0^y \tilde{G}(u+y-t) \, dt \to \infty$, we obtain (4.1). When $x \to \infty$, $F_1(x; \delta, D|u) \to ((1 + \beta)/\beta) \tilde{K}(u) - (1/\beta) \tilde{K} \ast \tilde{H}(u)$. Eqs. (4.2) and (4.3) can be shown from (2.9) and (2.10), respectively, by similar arguments. \qed

**Corollary 6.** For $D > 0$, the discounted defective probability density function of $U(T-)$ is

$$f_1(x; \delta, D|u) = \begin{cases} \lambda e^{-\rho x} \tilde{P}(x) \frac{e^{\rho u} - \tilde{K}(u+y)- \rho \int_0^u e^{\rho t} \tilde{K}(u-t) \, dt}{c + 2\rho D} \gamma(0), & \text{if } 0 \leq u < x, \\
\lambda e^{-\rho x} \tilde{P}(x) \frac{e^{\rho u} - \tilde{K}(u+y)- \rho \int_0^u e^{\rho t} \tilde{K}(u-t) \, dt}{c + 2\rho D} \gamma(0), & \text{if } 0 < x \leq u \end{cases} \tag{4.4}$$

with $f_1(x; \delta, D|0) = 0$.

For $D = 0$,

$$f_1(x; \delta, 0|u) = \begin{cases} \frac{\lambda}{c} e^{-\rho x} \tilde{P}(x) \frac{e^{\rho u} - \tilde{K}_0(u)- \rho \int_0^u e^{\rho t} \tilde{K}_0(u-t) \, dt}{1 - \tilde{K}_0(0)}, & \text{if } 0 \leq u < x, \\
\frac{\lambda}{c} e^{-\rho x} \tilde{P}(x) \frac{e^{\rho u} - \tilde{K}_0(u)- \rho \int_0^u e^{\rho t} \tilde{K}_0(u-t) \, dt}{1 - \tilde{K}_0(0)}, & \text{if } 0 < x \leq u \end{cases} \tag{4.5}$$

with $f_1(x; \delta, 0|0) = (\lambda/c)e^{-\rho x} \tilde{P}(x)$.

For $D = 0$ and $\delta = 0$,

$$f_1(x; 0, 0|u) = \begin{cases} \frac{\lambda}{c} \tilde{P}(x) \frac{1 - \psi_0(u)}{1 - \psi_0(0)}, & \text{if } 0 \leq u < x, \\
\frac{\lambda}{c} \tilde{P}(x) \frac{1 - \psi_0(u)}{1 - \psi_0(0)}, & \text{if } 0 < x \leq u, \end{cases} \tag{4.6}$$

which is (3.1) and (3.2) of Dickson (1992) with $f_1(x; 0, 0|0) = (\lambda/c) \tilde{P}(x)$. 

Proof. By (2.14) and $\tilde{K}(0) = 1/(1 + \beta)$, differentiating (4.1) with respect to $x$ leads to (4.4) for $0 \leq u < x$. If $0 < x \leq u$, by (2.14) and $[K \ast H(u)]' = b[K \ast H(u) - \tilde{K}(u)]$, we have

\[
\begin{align*}
    f_1(x; \delta, D|u) & = \frac{1}{\beta} \left[ G'(x) + \frac{b\rho}{a} \tilde{\Gamma}(x) \right] \tilde{K}(u-x) + \frac{1}{\beta} \left[ \tilde{G}'(x) - \frac{b}{a} \tilde{\Gamma}'(x) \right] \overline{K} \ast H(u-x) \\
    & + \frac{b}{\beta} \left[ \tilde{G}(x) - \frac{b}{a} \tilde{\Gamma}(x) \right] [K \ast H(u-x) - \tilde{K}(u-x)] \\
    & - \frac{\lambda}{c + 2\rho D} e^{-\rho u} \tilde{P}(x) \frac{K \ast H(u) + \rho \int_0^u e^{\rho t} \tilde{K}(u-t) \, dt}{1 - \tilde{K}(0)} \\
    & = \frac{1}{\beta} \left[ G'(x) + \frac{b\rho}{a} \tilde{\Gamma}(x) - b\tilde{G}(x) + \frac{b^2}{a} \tilde{\Gamma}(x) \right] \tilde{K}(u-x) \\
    & - \frac{1}{\beta} \left[ G'(x) - \frac{b}{a} \tilde{\Gamma}'(x) - b\tilde{G}(x) + \frac{b^2}{a} \tilde{\Gamma}(x) \right] \overline{K} \ast H(u-x) \\
    & - \frac{\lambda}{c + 2\rho D} e^{-\rho u} \tilde{P}(x) \frac{K \ast H(u) + \rho \int_0^u e^{\rho t} \tilde{K}(u-t) \, dt}{1 - \tilde{K}(0)}.
\end{align*}
\]

From $G(x) = \tilde{\Gamma}(x) + G'(x)/b$ and $a = b + \rho$, the first term on the right side vanishes, and the second term becomes $(1/\beta)(b/a)|\rho \tilde{\Gamma}(x) + \Gamma'(x)|K \ast H(u-x) = (\lambda/(c + 2\rho D))(1 + \beta)/(\beta) \tilde{P}(x) K \ast H(u-x)$ by (2.14). Combining this with the third term gives (4.4) for $0 < x \leq u$.

Similar arguments yield (4.5) from (4.2). If $\delta = 0$, then $\tilde{K}_0(u)|_{\delta = 0} = \psi_0(u)$ and (4.5) reduces to (4.6). \qed

We remark that if we equate (6.5) and (6.6) of Gerber and Shiu (1998) and (4.5), we have

\[
\tilde{K}_{\rho_0}(u) = e^{\rho_0 u} - \frac{1 - \tilde{K}_{\rho_0}(0)}{1 - \tilde{K}(0)} \left[ e^{\rho_0 u} - \tilde{K}_0(u) - \rho_0 \int_0^u e^{\rho_0 t} \tilde{K}_0(u-t) \, dt \right],
\]

where $\tilde{K}_{\rho_0}(u)$ is defined as $\tilde{K}_{\rho_0}(u) = E[e^{-\beta T - \rho_0 U(T)} I(T < \infty)|U(0) = u]$, $u \geq 0$ with $\tilde{K}_{\rho_0}(u)|_{\delta = 0} = \tilde{K}_0(u)|_{\delta = 0} = \psi_0(u)$.

**Theorem 4.** When $0 \leq u < x$, the discounted defective distribution function of $U(T-)$ for $D > 0$ satisfies the defective renewal equation

\[
F_1(x; \delta, D|u) = \frac{1}{1 + \beta} \int_0^u F_1(x; \delta, D|u-y) \, dG(y) + \frac{1}{1 + \beta} \left\{ \tilde{\Gamma} \ast H(u) - \frac{b}{a} \beta e^{-\rho x} \tilde{\Gamma}(x) [e^{\rho u} - e^{-\beta u}] \right\}.
\]

For $D = 0$,

\[
F_1(x; \delta, 0|u) = \frac{1}{1 + \beta_0} \int_0^u F_1(x; \delta, 0|u-y) \, d\Gamma_0(y) + \frac{1}{1 + \beta_0} [\tilde{\Gamma}_0(u) - e^{-\rho_0(x-u)} \tilde{\Gamma}_0(x)].
\]

For $D = 0$ and $\delta = 0$,

\[
F_1(x; 0, 0|u) = \frac{1}{1 + \delta} \int_0^u F_1(x; 0, 0|u-y) \, dP_1(y) + \frac{1}{1 + \delta} [\tilde{P}_1(u) - \tilde{P}_1(x)].
\]

**Proof.** For any fixed $x$, let

\[
w(z, y) = \begin{cases} 1, & \text{if } z \leq x, \\ 0, & \text{otherwise.} \end{cases}
\]
Then by (1.6) and (1.7), \( \phi_w(u) \) in (1.5) turns out to be \( \phi_w(u) = \int_0^u f_z f_s e^{-\beta t} f(z, y, t; D|u) \frac{dz}{y} \), the discounted distribution function of \( U(T-) \).

By Tsai and Willmot (2001), \( F_1(x; \delta, D|u) \) satisfies the defective renewal equation

\[
F_1(x; \delta, D|u) = \int_0^u F'(x; \delta, D|u - y) g(y) \frac{dy}{y} + \frac{c}{D} \int_0^u e^{-b(u - y)} \gamma_0(s) \frac{ds}{y}, \tag{4.10}
\]

where \( \gamma_0(s) = (\lambda/c) \int_0^\infty e^{-\rho(z-s)} \int_0^\infty w(z, y) p(z + y) \frac{dy}{y} \frac{dz}{z} \). Then by the definition of \( w(z, y) \),

\[
\gamma_0(s) = \frac{\lambda}{c} \int_0^\infty e^{-\rho(z-s)} \cdot \frac{\rho(s) \int_0^\infty e^{-\rho(z-s)} P(z) \frac{dz}{z}}{z} \frac{dz}{z}.
\]

From (1.13), \( \gamma_0(s)/\rho(s) \int_0^\infty e^{-\rho y} P(y) \frac{dy}{y} = \tilde{F}(s) - e^{-\rho(x-s)} \tilde{F}(x) \). Combining this with (1.12) leads to

\[
\frac{(c/D) \lambda}{\rho} \int_0^\infty e^{-\rho(x-y)} \gamma_0(s) \frac{ds}{y} = \int_0^u H'(x-y)[\tilde{F}(s)-e^{-\rho(x-s)} \tilde{F}(x)] \frac{ds}{s} = \tilde{G} \ast H(u) - \frac{b}{a} e^{-\rho x} \tilde{P}(x)[e^{\rho u} - e^{-bu}] + \frac{\lambda \rho}{c + 2 \rho D} e^{-\rho x} \tilde{P}(x).
\]

Since \( \int_0^\infty g(z) \frac{dz}{z} = 1/(1 + \beta) \) and \( G'(x) = g(x) \int_0^\infty g(z) \frac{dz}{z} \), multiplying the right side of (4.10) by \((1/(1 + \beta)) / \int_0^\infty g(z) \frac{dz}{z} \) leads to (4.7).

Eqs. (4.8) and (4.9) can be shown by similar arguments. Alternatively, when \( D = 0 \), then \( \beta = \beta_0, \rho = \rho_0, b/a = 1, \tilde{G} \ast H(u) = \tilde{G}_0(u) \) and \( \tilde{H}(u) = e^{-bu} = 0 \) for \( u > 0 \), which implies that (4.7) turns out to be (4.8) for \( u > 0 \). If \( \delta = 0 \), then \( \beta_0 = \theta, \rho_0 = 0 \) and \( \tilde{G}_0(x) = P_1(x) \), (4.8) reduces to (4.9).

**Corollary 7.** When \( 0 < u < x \), the discounted defective probability density function of \( U(T-) \) for \( D > 0 \) satisfies the defective renewal equation

\[
f_1(x; \delta, D|u) = \frac{1}{1 + \beta} \int_0^u f_1(x; \delta, D|u - y) \frac{dy}{y} + \frac{\lambda}{c + 2 \rho D} e^{-\rho x} \tilde{P}(x)[e^{\rho u} - e^{-bu}]. \tag{4.11}
\]

For \( D = 0 \),

\[
f_1(x; \delta, 0|u) = \frac{1}{1 + \beta_0} \int_0^u f_1(x; \delta, 0|u - y) \frac{dy}{y} + \frac{\lambda}{c} e^{-\rho_0(x-u)} \tilde{P}(x). \tag{4.12}
\]

For \( D = 0 \) and \( \delta = 0 \),

\[
f_1(x; 0, 0|u) = \frac{1}{1 + \theta} \int_0^u f_1(x; 0, 0|u - y) \frac{dy}{y} + \frac{\lambda}{c} \tilde{P}(x). \tag{4.13}
\]

**Proof.** Differentiating (4.7)–(4.9) with respect to \( x \) leads to (4.11)–(4.13), respectively, with the help of (2.14).

Dickson (1992) proposed a relationship between \( F_1(x; 0, 0|u) \), the probability that ruin with initial surplus \( u \) and the surplus immediately prior to ruin caused by a claim is at most \( x \), and \( F_2(y; 0, 0|u) \), the probability of ruin with initial surplus \( u \) and the deficit immediately after the claim causing ruin is at most \( y \), as follows:

\[
F_1(x; 0, 0|u) = F_2(x; 0, 0|u - x) - \left[ 1 + \frac{\tilde{P}(x)}{\theta} \right] \left[ \psi_0(u - x) - \psi_0(u) \right], \quad 0 < x \leq u. \tag{4.14}
\]

We have corresponding relationships between \( F_1(x; \delta, D|u) \) and \( F_2(y; \delta, D|u) \) and between \( F_1(x; \delta, 0|u) \) and \( F_2(y; \delta, 0|u) \) as well.

**Lemma 1.** When \( 0 < x \leq u \), for \( D > 0 \),
Proof. By (3.1),

$$F_1(x; \delta, D|u) = F_2(x; \delta, D|u - x) - [\bar{K}(u - x) - \bar{K}(u)] - \frac{1}{\beta} e^{-\rho x \Gamma(x)} \int_0^x e^{\rho t \bar{K}(u - t)} \, dt$$

$$- \frac{1}{\beta} \left[ 1 - \frac{b}{a} \bar{F}(x) \right] [\bar{K}(u - x) - \bar{K} \ast \bar{H}(u - x)]$$

$$+ \frac{1}{\beta} \left[ 1 - \frac{b}{a} e^{-\alpha x \bar{F}(x)} \right] [\bar{K}(u) - \bar{K} \ast \bar{H}(u)].$$

(4.15)

For \(D = 0\),

$$F_1(x; \delta, 0|u) = F_2(x; \delta, 0|u - x) - [\bar{K}_0(u - x) - \bar{K}_0(u)] - \frac{1}{\beta_0} e^{-\rho_0 x \bar{F}_0(x)} \int_0^x e^{\rho_0 t \bar{K}_0(u - t)} \, dt. \quad (4.16)$$

For \(D = 0\) and \(\delta = 0\), (4.16) reduces to (4.14).

**Proof.** By (3.1), $F_2(x; \delta, D|u - x) = ((1 + \beta)/\beta) [\bar{K}(u - x) - \bar{K}(u)] - (1/\beta) G(y) \bar{K} \ast \bar{H}(u - x) + (1/\beta) \int_0^y \bar{K}(u - t) \, dG(t)$. Deducting this from the expression in (4.1) for \(0 < x \leq u\) gives

$$F_1(x; \delta, D|u) = F_2(x; \delta, D|u - x) - \frac{1}{\beta} \left[ \bar{K}(u - x) - \bar{K}(u) \right] + \frac{1}{\beta} \left[ \bar{K} \ast \bar{H}(u - x) - \bar{K} \ast \bar{H}(u) \right]$$

$$- \frac{1}{\beta} \frac{b}{a} \bar{F}(x) \bar{K} \ast \bar{H}(u - x) + \frac{1}{\beta} \frac{b}{a} e^{-\alpha x \bar{F}(x)} \left[ \rho \int_0^x e^{\rho t \bar{K}(u - t)} \, dt + \bar{K} \ast \bar{H}(u) \right].$$

By integration by parts, the last term on the right side becomes

$$(1/\beta) (b/a) e^{-\alpha x \bar{F}(x)} \left[ e^{\rho x \bar{K}(u - x)} - \bar{K}(u) \right] - \int_0^x e^{\rho t \bar{K}(u - t)} \, dt + \bar{K} \ast \bar{H}(u).$$

Then (4.15) follows after some algebra.

When \(D = 0\), then \(b/a = 1, \beta = \beta_0, \rho = \rho_0\), and both \(\bar{K}(u)\) and \(\bar{K} \ast \bar{H}(u)\) are equal to \(\bar{K}_0(u)\), implying that (4.15) becomes (4.16). If \(\delta = 0\), then \(\beta_0 = \theta, \rho_0 = 0, \bar{F}_0(x) = \bar{P}_1(x) \bar{K}_0(u)\) at \(\delta = 0\), implying that (4.14) from (4.16).

**5. Discounted distribution and probability density functions of \([U(T) - |U(T)|]\)**

When ruin occurs due to a claim, \([U(T) - |U(T)|]\) is the amount of the claim causing ruin. Examining Corollaries 1 and 6 gives the following:

$$f(x, y; D|u) = f(x, y; \delta, D|u) = \frac{f(x, y; \delta, 0|u)}{f_1(x; \delta, 0|u)} = \frac{f(x, y; 0, 0|u)}{f_1(x; 0, 0|u)} = \frac{p(x + y)}{P(x)}, \quad (5.1)$$

independent of \(u\), where \(x + y\) is the amount of the claim causing ruin. Eq. (5.1) can be obtained directly by Bayes’ theorem: the (discounted) joint probability density function of \(U(T) - |U(T)|\) at \((x, y)\) divided by the (discounted) marginal probability density function of \(U(T) - |U(T)|\) at \(x\) is equal to the (discounted) conditional probability density function of \([U(T)| y]\), which is \(p(x + y) \int_0^\infty p(x + y) \, dy = p(x + y) \bar{P}(x)\) (see Dufresne and Gerber, 1988; Gerber and Shiu, 1997, 1998). Dickson and Egidio dos Reis (1994) showed (5.1) for the case \(D = 0\) and \(\delta = 0\) by an alternative approach.

In the case that there is no ruin due to a claim, \(p(x + y)\) is just the probability density function of the individual claim amount; therefore, the random variable \([U(T) - |U(T)|]\) is undefined and corresponding probability density function is meaningless.

The amount of claim causing ruin has a different distribution than the amount of an arbitrary claim (since it is large enough to cause ruin). Many others have discussed its distribution in the classical model (see also Dickson, 1992, 1993), so it is clearly of interest. In addition, studying the distribution of \([U(T) - |U(T)|]\) helps one understand the risk of ruin. To derive the discounted probability density function of \([U(T) - |U(T)|]\), let \(w(x, y) = \)
\[ x + y, \text{ then by (1.6), } \phi_w(u) \text{ in (1.5) turns out to be } \phi_w(u) = \int_0^\infty \int_0^\infty e^{-\delta t} (x + y) f(x, y, t; D|u) \, dx \, dy = \int_0^\infty \delta f_Z(z; \delta, D|u) \, dz, \]

the expectation of the amount of the claim causing ruin, where

\[ f_Z(z; \delta, D|u) = \int_0^z f(x, z - x; \delta, D|u) \, dx = p(z) \int_0^z \frac{f_1(x; \delta, D|u)}{P(x)} \, dx \quad (5.2) \]

by (5.1), is the discounted defective probability density function of the amount of the claim causing ruin. Eq. (5.2) provides an alternative formula for \( f_Z(z; \delta, D|u) \) since explicit expression for \( f(x, y; \delta, D|u) \) and \( f_1(x; \delta, D|u) \) is available from (2.11) and (4.4), respectively.

**Theorem 5.** For \( D > 0 \), the discounted defective probability density function of \([U(T) - |U(T)|] \) is

\[
f_Z(z; \delta, D|u) = \begin{cases} 
\frac{\lambda}{c+2\rho D} \frac{e^{-\rho z} p(z)}{1-K(0)} \left\{ \int_0^u K \ast H(u-t) \, dt - \int_0^u \tilde{K}(u-t) \, dt \right\}, & \text{if } 0 \leq u < z, \\
\frac{\lambda}{c+2\rho D} \frac{e^{-\rho z} p(z)}{1-K(0)} \left\{ \int_0^z K \ast H(u-t) \, dt - \int_0^z \tilde{K}(u-t) \, dt \right\}, & \text{if } 0 < z \leq u 
\end{cases}
\]

with \( f_Z(z; \delta, D|0) = 0 \).

For \( D = 0 \),

\[
f_Z(z; \delta, 0|u) = \begin{cases} 
\frac{\lambda}{c\rho_0} \frac{e^{-\rho_0 z} p(z)}{1-K(0)} \left\{ [e^{\rho_0 z} - e^{\rho_0 u}] - [e^{\rho_0 z} - 1] \tilde{K}_0(u) + \rho_0 \int_0^u e^{\rho t} \tilde{K}_0(u-t) \, dt \right\}, & \text{if } 0 \leq u < z, \\
\frac{\lambda}{c\rho_0} \frac{e^{-\rho_0 z} p(z)}{1-K(0)} \left\{ -[e^{\rho_0 z} - 1] \tilde{K}_0(u) + \rho_0 \int_0^z e^{\rho t} \tilde{K}_0(u-t) \, dt \right\}, & \text{if } 0 < z \leq u 
\end{cases}
\]

with \( f_Z(z; \delta, 0|0) = (\lambda/c)((1 - e^{-\rho_0 z})/\rho_0) p(z) \).

For \( D = 0 \) and \( \delta = 0 \),

\[
f_Z(z; 0, 0|u) = \begin{cases} 
\frac{\lambda}{c} \frac{p(z)}{1 - \psi_0(0)} \left\{ [z-u] - \psi_0(u)z + \int_0^u \psi_0(u-t) \, dt \right\}, & \text{if } 0 \leq u < z, \\
\frac{\lambda}{c} \frac{p(z)}{1 - \psi_0(0)} \left\{ -\psi_0(u)z + \int_0^z \psi_0(u-t) \, dt \right\}, & \text{if } 0 < z \leq u 
\end{cases}
\]

with \( f_Z(z; 0, 0|0) = (\lambda/c)zp(z) \), which is (3.2) of Dickson (1993).

**Proof.** By (4.4), if \( 0 < z \leq u \), (5.2) becomes

\[
f_Z(z; \delta, D|u) = \frac{\lambda}{c+2\rho D} \frac{p(z)}{1-K(0)} \	imes \left\{ \int_0^z K \ast H(u-x) \, dx - K \ast H(u) \int_0^z e^{-\rho x} \, dx - \rho \int_0^z e^{-\rho x} \int_0^x e^{\rho t} \tilde{K}(u-t) \, dt \, dx \right\}.
\]
Since \( \int_0^\infty e^{-px} f_0^x e^{pt} \bar{K}(u-t) \, dt \, dx = \int_0^\infty \int_0^x e^{-pt} \bar{K}(u-t) \, dx \, dt \) by changing the order of integration, the right side becomes

\[
\frac{\lambda}{c + 2\rho D} \frac{p(z)}{1 - K(0)} \left\{ \int_0^z K * H(u-x) \, dx - \frac{1 - e^{-\rho z}}{\rho} K * H(u) - \int_0^z [1 - e^{-\rho(z-t)}] \bar{K}(u-t) \, dt \right\}. \tag{5.6}
\]

Then the expression in (5.3) for \( 0 < z \leq u \) follows after some algebra.

If \( 0 \leq u < z \), decompose \( f_0^z \) into \( f_0^u \) and \( f_u^z \). Then from (5.6)

\[
p(z) \int_0^u f_1(x; \delta, D|u) \frac{1}{P(x)} \, dx = \frac{\lambda}{c + 2\rho D} \frac{p(z)}{1 - K(0)} \left\{ \int_0^u K * H(u-t) \, dt - \frac{1 - e^{-\rho u}}{\rho} K * H(u) - \int_0^u [1 - e^{-\rho(u-t)}] \bar{K}(u-t) \, dt \right\},
\]

and from (4.4)

\[
p(z) \int_u^z f_1(x; \delta, D|u) \frac{1}{P(x)} \, dx = \frac{\lambda}{c + 2\rho D} \frac{p(z)}{1 - K(0)} \left\{ \frac{e^{\rho u} - K * H(u)}{\rho} - \int_0^u e^{pt} \bar{K}(u-t) \, dt \right\} \int_u^z e^{-px} \, dx
\]

\[
= \frac{\lambda}{c + 2\rho D} \frac{p(z)}{1 - K(0)} \left\{ \frac{1 - e^{-\rho(z-u)}}{\rho} - \frac{e^{-\rho u} - e^{-\rho z}}{\rho} K * H(u)
\right\} - \int_0^u [e^{-\rho(u-t)} - e^{-\rho(z-t)}] \bar{K}(u-t) \, dt \right\}.
\]

Therefore, combining these two expressions above yields (5.3) for \( 0 \leq u < z \) after some algebra.

Similar arguments lead to (5.4) and (5.5) from (4.5), (4.6) and (5.2). Alternatively, when \( D = 0, \rho = \rho_0, \overline{K * H}(u) = \overline{K_0}(u) \) for \( u > 0 \), and \( \overline{K}(u) = \overline{K_0}(u) \), which implies that (5.3) reduces to (5.4) for \( u > 0 \). When \( \delta = 0 \), then \( \rho_0 = 0 \) and \( \overline{K_0}(u)|_{u=0} = \psi_0(u) \), one has that (5.4) implies (5.5) by L’Hospital’s rule. \( \square \)

The expressions for \( F_Z(z; \delta, D|u) \), \( F_Z(z; \delta, 0|u) \) and \( F_Z(z; 0, 0|u) \), the discounted defective distribution functions of the amount of the claim causing ruin, can be obtained from (5.2) as follows:

\[
F_Z(z; \delta, D|u) = \int_0^z p(y) \int_0^y f_1(x; \delta, D|u) \frac{1}{P(x)} \, dx \, dy = F_1(z; \delta, D|u) - \bar{P}(z) \int_0^z f_1(x; \delta, D|u) \frac{1}{P(x)} \, dx \tag{5.7}
\]

by integration by parts, i.e., subtraction of (5.3)–(5.5) with \( p(z) \) replaced with \( \bar{P}(z) \) (since \( f_Z(z; \delta, D|u) = p(z) \int_0^z f_1(x; \delta, D|u) / P(x) \, dx \) from (4.1)–(4.3) with \( x \) replaced with \( z \) gives \( F_Z(z; \delta, D|u) \), \( F_Z(z; \delta, 0|u) \) and \( F_Z(z; 0, 0|u) \), respectively.

In addition, integration of \( f_Z(t; \delta, D|0) \), \( f_Z(t; \delta, 0|0) \) and \( f_Z(t; 0, 0|0) \) from \( t = 0 \) to \( t = z \) in (5.3)–(5.5) for \( u = 0 \), respectively, yields

\[
F_Z(z; \delta, 0|0) = \frac{\lambda}{c} \left[ \int_0^z e^{-\rho_0 t} \bar{P}(t) \, dt - \frac{1 - e^{-\rho_0 z}}{\rho_0} \bar{P}(z) \right] \quad F_Z(z; \delta, D|0) = 0,
\]

and

\[
F_Z(z; 0, 0|0) = \frac{\lambda}{c} [p_1 P_1(z) - z \bar{P}(z)],
\]

which is (3.1) of Dickson (1993).
We remark that from (5.7)
\[
\lim_{z \to \infty} F_Z(z; \delta, D|u) = \lim_{x \to \infty} F_1(x; \delta, D|u) = \lim_{y \to \infty} F_2(y; \delta, D|u) = \frac{1+\beta}{\beta} \bar{K}(u) - \frac{1}{\beta} \overline{K \ast H}(u),
\]
\[
\lim_{z \to \infty} F_Z(z; \delta, D|0) = \lim_{x \to \infty} F_1(x; \delta, 0|u) = \lim_{y \to \infty} F_2(y; \delta, 0|u) = \bar{K}_0(u),
\]
and
\[
\lim_{z \to \infty} F_Z(z; 0, 0|u) = \lim_{x \to \infty} F_1(x; 0, 0|u) = \lim_{y \to \infty} F_2(y; 0, 0|u) = \psi_0(u).
\]

**Theorem 6.** When \(0 \leq u < z\), the discounted defective distribution function of \([U(T) - |U(T)|]\) for \(D > 0\) satisfies the defective renewal equation
\[
F_Z(z; \delta, D|u) = \frac{1}{1+\beta} \int_0^u F_Z(z; \delta, D|u-y) \, dG(y) + \frac{1}{1+\beta} \left\{ \bar{F} \ast H(u) - \frac{b}{a} e^{-\rho z} \bar{F}(z) [e^{\beta u} - e^{-bu}] \right\}
- \frac{\lambda}{c + 2\rho D} \bar{P}(z) [e^{\beta u} - e^{-bu}].
\]
(5.8)

For \(D = 0\),
\[
F_Z(z; \delta, D|u) = \frac{1}{1+\beta_0} \int_0^u F_Z(z; \delta, D|0|u-y) \, d\Gamma_0(y) + \frac{1}{1+\beta_0} \left[ \bar{F}_0(u) - e^{-\rho_0(z-u)} \bar{F}_0(z) \right]
- \frac{\lambda}{c} \frac{e^{\beta_0 u} (1 - e^{-\rho_0 z})}{\rho_0} \bar{P}(z).
\]
(5.9)

For \(D = 0\) and \(\delta = 0\),
\[
F_Z(z; 0, 0|u) = \frac{1}{1+\theta} \int_0^u F_Z(z; 0, 0|u-y) \, dP_1(y) + \frac{1}{1+\theta} \left[ \bar{P}_1(u) - \bar{P}_1(z) - \frac{z}{p_1} \bar{P}(z) \right].
\]
(5.10)

**Proof.** From (5.7),
\[
\int_0^u F_Z(z; D|u-y) \, dG(y) = \int_0^u F_1(z; D|u-y) \, dG(y) - \bar{P}(z) \int_0^z \frac{f_1(x; D|u-y)}{P(x)} \, dx \, dG(y).
\]
Subtraction of this equation from (5.7) yields
\[
F_Z(z; \delta, D|u) - \frac{1}{1+\beta} \int_0^u F_Z(z; \delta, D|u-y) \, dG(y) = F_1(z; \delta, D|u) - \frac{1}{1+\beta} \int_0^u F_1(z; \delta, D|u-y) \, dG(y)
- \bar{P}(z) \int_0^z \frac{1}{P(x)} \left[ f_1(x; \delta, D|u) - \frac{1}{1+\beta} \int_0^u f_1(x; \delta, D|u-y) \, dG(y) \right] \, dx.
\]
Then (5.8) is easily obtained by (4.7) and (4.11) after some algebra, and (5.9) and (5.10) can be shown by similar arguments. Alternatively, when \(D = 0\), then \(\beta = \beta_0, \rho = \rho_0, b/a = 1, \bar{F} \ast H(u) = \bar{F}_0(u)\) and \(H(u) = e^{-bu} = 0\) for \(u > 0\), which implies that (5.8) simplifies to (5.9) for \(u > 0\). If \(\delta = 0\), then \(\beta_0 = \theta, \rho_0 = 0\) and \(\bar{F}_0(x) = P_1(x)\), (5.9) reduces to (5.10).

**Corollary 8.** When \(0 \leq u < z\), the discounted defective probability density function of \([U(T) - |U(T)|]\) for \(D > 0\) satisfies the defective renewal equation
\[
f_Z(z; \delta, D|u) = \frac{1}{1+\beta} \int_0^u f_Z(z; \delta, D|u-y) \, dG(y) + \frac{\lambda}{c + 2\rho D} \frac{1-e^{-\rho z}}{\rho} \bar{P}(z) [e^{\beta u} - e^{-bu}].
\]
(5.11)
For $D = 0$,

$$f_{Z}(z; \delta, 0|u) = \frac{1}{1 + \rho_0} \int_{0}^{u} f_{Z}(z; \delta, 0|u - y) \, dP_0(y) + \frac{\lambda e^{\rho_0 u}}{c} \frac{1 - e^{-\rho_0 u}}{\rho_0} p(z). \quad (5.12)$$

For $D = 0$ and $\delta = 0$,

$$f_{Z}(z; 0, 0|u) = \frac{1}{1 + \theta} \int_{0}^{u} f_{Z}(z; 0, 0|u - y) \, dP_1(y) + \frac{\lambda}{c} z p(z). \quad (5.13)$$

**Proof.** Differentiating (5.8)–(5.10) with respect to $z$ gives (5.11)–(5.13), respectively, with the help of (2.14). \(\Box\)

Since (4.15) gives an expression for $F_{1}(z; \delta, D|u) - F_{2}(z; \delta, D|u - z)$ for $0 < z \leq u$, combining this with (5.7), $F_{Z}(z; \delta, D|u) = F_{1}(z; \delta, D|u) - \bar{P}(z) \int_{0}^{z} f_{1}(x; \delta, D|u/\bar{P}(x)) \, dx$, we can also obtain an expression for $F_{Z}(z; \delta, D|u) - F_{2}(z; \delta, D|u - z)$ for $0 < z \leq u$.

Finally, we remark that since each of the expressions for the discounted probability distribution functions $F(x, y; \delta, D|u)$, $F_{1}(x; \delta, D|u)$, $F_{2}(x; \delta, D|u)$ and $F_{Z}(z; \delta, D|u)$, and for the discounted probability density functions $f(x, y; \delta, D|u)$, $f_{1}(x; \delta, D|u)$, $f_{2}(y; \delta, D|u)$ and $f_{Z}(z; \delta, D|u)$ involve $\bar{G}(u)$, $\bar{P}(u)$, $K(u)$ and $K \ast H(u)$ which have been shown by Tsai (2001) and Tsai and Willmot (2001) to have explicit analytical solutions if $P(x)$ is a combination of exponentials or a mixture of Erlangs, each of these expressions for the discounted distribution functions and probability density functions can be obtained explicitly if $P(x)$ is a combination of exponentials or a mixture of Erlangs.

**References**


