Ordering risks: Expected utility theory versus Yaari’s dual theory of risk

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Abstract

We introduce a class of partial orderings of risks that are dual to stochastic dominance orderings. These arise as “distortion-free” orderings in Yaari’s dual theory of risk (1987). We show that these dual orderings are equivalent to inverse stochastic dominance orderings (Muliere and Scarsini, 1989). We motivate third dual stochastic dominance via insurance economics, while providing an alternative interpretation for second (dual) stochastic dominance. We apply dual stochastic dominance to actuarial science and show how the dual ordering of risks is related to ordering income distributions in the economics of income inequality. © 1998 Elsevier Science B.V. All rights reserved

Keywords: Stochastic dominance; Inverse stochastic dominance; Partial orderings; Insurance economics; Income inequality; Gini index

1. Economic theories of risk and uncertainty

Insurance is an enterprise for managing risk and uncertainty. Economic theories of risk and uncertainty are essential parts of insurance economics; see, for example, Borch (1968). Expected utility theory has greatly contributed to understanding the economics of risk and uncertainty for the past several decades. For example, expected utility theory has been used to determine: (i) optimal (re)insurance policies, as in Arrow’s theorem (Arrow, 1971; van Heerwaarden, 1991; Taylor, 1992a, b; Kass et al., 1994), (ii) optimal insurance policies in the presence of adverse selection (Rothschild and Stiglitz, 1976; Young and Browne, 1997a) or moral hazard (Shavell, 1979), and (iii) optimal insurance purchases versus precautionary saving (Yaari, 1965), to name but three important applications of utility theory in insurance economics. Expected utility theory has also been linked to stochastic dominance because a (higher) stochastic dominance ordering can be shown to be equivalent to the common, partial ordering determined by a class of utility functions.

In the mid-1980s, Yaari (1987) developed a parallel theory of risk by modifying the independence axiom of von Neumann and Morgenstern (1947). In Yaari’s theory, attitudes toward risks are characterized by a distortion applied to probability distribution functions, in contrast to expected utility theory in which attitudes toward risks are characterized by a utility function of wealth. Yaari’s dual theory is relatively new and, thus, not widely applied in insurance economics. In this paper, we study the implied orderings of risks in Yaari’s dual theory; these partial

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ordering may be considered dual to stochastic dominance orderings. In other work, we show how to price insurance
risks in Yaari’s dual theory (Wang, 1996a; Wang and Young, 1997; Wang et al., 1997). Wang and Dhaene (1997)
and Dhaene et al. (1997) determine maximal aggregate stop-loss premiums for a given collection of risks; their
work relies on Yaari’s dual theory, and it complements the work in this paper. Young and Browne (1997b) determine
optimal insurance contracts under adverse selection in the framework of Yaari’s theory.

In Section 2, we review the axioms for rational decision making that comprise expected utility theory. We, then,
review how Yaari’s theory can be viewed as dual to expected utility theory. In Section 3, we remind the reader
how classes of utility functions can be used to order risks in a utility-free way and state a result that pairs a given
utility-free ordering with the one defined by a (higher) stochastic dominance.

In Section 4, we present the dual to utility-free orderings in Yaari’s theory and develop distortion-free orderings
of risks. We provide a motivation from insurance economics for third dual stochastic dominance, while also providing
an alternative interpretation for second (dual) stochastic dominance. Our main result is that a given distortion-free
ordering is equivalent to the one defined by a (higher) dual stochastic dominance. As corollaries to our main result,
we show that first and second stochastic dominance are equivalent to first and second dual stochastic dominance,
respectively. However, the equivalence does not hold higher-order stochastic dominance; namely, third stochastic
dominance differs from third dual stochastic dominance.

In Section 5, we generalize a result of Wang and Dhaene (1997) in which they show that second stochastic
dominance is preserved under adding (comonotonic) risks. We obtain, as a special case of a result of van Heerwaarden
(1991), that optimal reinsurance in Yaari’s theory is stop-loss insurance. We also show how the dual moments of
a risk are related to measures of income inequality and suggest an additional application for third dual stochastic
dominance (Muliere and Scarsini, 1989).

2. Axiomatic foundations

Let \( X, Y, \) or \( Z \) represent a nonnegative random loss, or risk, and let \( S_X(t) = \Pr(X > t), t > 0, \) denote the
decumulative distribution function (ddf) of \( X. S_X \) is a nonincreasing function from \( \mathbb{R}^+ \) to \([0, 1]\) with inverse \( S_X^{-1} \)
defined by

\[
S_X^{-1}(q) = \inf\{t \geq 0: S_X(t) \leq q\}, \quad 0 \leq q \leq 1,
\]

with \( S_X^{-1}(0) = +\infty, \) if \( S_X(t) > 0 \) for all \( t \geq 0. \) Note that the inverse ddf is a nonincreasing function from \([0, 1]\) to \( \mathbb{R}^+. \)

Let the symbol \( \prec \) represent (not necessarily strict) riskiness; i.e., \( X \prec Y \) means that \( X \) is less risky than \( Y. \) von
Neumann and Morgenstern (1947) propose axioms for rationally ordering risks. As presented in Yaari (1987), they
are:

\begin{enumerate}
  \item \textbf{EU1} If risks \( X \) and \( Y \) have the same decumulative distribution functions, then \( X \) and \( Y \) are equally risky. Specifically,
  if \( S_X = S_Y, \) then \( X \prec Y \) and \( Y \prec X. \)
  \item \textbf{EU2} \( \prec \) is reflexive, transitive and connected.
  \item \textbf{EU3} \( \prec \) is continuous in the topology of weak convergence.
  \item \textbf{EU4} If \( S_X \preceq S_Y, \) then \( X \prec Y. \)
  \item \textbf{EU5} If \( X \prec Y, \) and \( Z \) is any risk, then \((p, X), (1 - p, Z) \prec (p, Y), (1 - p, Z), \) for all \( p \in [0, 1], \) in which
  \[ (p, X), (1 - p, Z) \] is defined by the probabilistic mixture
  \[ S((p, X), (1 - p, Z))(t) = pS_X(t) + (1 - p)S_Z(t), \quad \text{for all } t \geq 0, \]
  and similarly for \((p, Y), (1 - p, Z).\)
\end{enumerate}

To put Axiom EU5 in the context of ordering insurance risks, suppose a decision maker (d.m.) for example, an
underwriter employed by an insurance company – is asked to choose between the following two options: Option 1
is to select a policyholder at random from Market 1 in which risks \( X \) make up proportion \( p \) of the market with risks
Z making up the rest of the market. Option 2 is to select a policyholder at random from Market 2 in which risks Y make up proportion $p$ of the market with risks $Z$ making up the rest. If the d.m. views risk $X$ as being less risky than $Y$, then Axiom EU5 implies that the d.m. will choose Option 1, for the same premium scheme.

From these five axioms of preference, von Neumann and Morgenstern (1947) show that there exists a utility function $u$ such that $X \prec Y$ iff their expected utilities satisfy $E[u(-X)] \geq E[u(-Y)]$, in which one defines a utility function and expected utility as follows:

**Definition 2.1.** A (normalized) utility function $u$ is a nondecreasing real-valued function on $\mathbb{R}$ with $u(0) = 0$. For a nonnegative random variable $X$, the expected utilities $E[u(X)]$ and $E[u(-X)]$ are given by

$$E[u(X)] = \int_0^{\infty} S_X(t) u(t) \, dt = \int_0^1 u[S_X^{-1}(q)] \, dq$$

and

$$E[u(-X)] = -E[\tilde{u}(X)],$$

where $\tilde{u}$ is the utility defined by $\tilde{u}(w) = -u(-w)$.

This axiomatic foundation of expected utility theory leads many to believe that utility theory is the only legitimate theoretical tool for decision making under uncertainty. We next present the axioms of Yaari's dual theory of risk, an alternative to the expected utility theory. Yaari (1987) and Röell (1987) consider random variables with bounded support in $[0, 1]$. They replace $S_X$ with its inverse $S_X^{-1}$ everywhere in the axioms of utility theory. Note that the first four axioms of utility are unchanged when one replace $S_X$ with $S_X^{-1}$. However, the fifth axiom becomes the following dual axiom.

**DU5** If $X \prec X$, if $Z$ is any risk, and if $p$ is any number in $[0, 1]$, then

$$W \prec V,$$

where $W$ and $V$ are the random variables with inverse ddf's given by $pS_X^{-1} + (1 - p)S_Y^{-1}$ and $pS_X^{-1} + (1 - p)S_Y^{-1}$, respectively.

**Definition 2.2.** As a conceptual extension of perfect correlation, $X$ and $Y$ are said to be comonotonic if there exists a risk $Z$ and nondecreasing real-valued functions $f$ and $h$ such that $X = f(Z)$ and $Y = h(Z)$.

In the context of insurance, it is usual to restrict the indemnity benefit, $I(x)$, to be a nondecreasing function of the underlying loss, $x$, in order to prevent a policyholder misrepresenting a loss downward. In this case, $X$ and $I(X)$ are comonotonic risks. Similarly, one often restricts the indemnity benefit to increase at a slower rate than the underlying loss in order to prevent a policyholder from misrepresenting a loss upward. Specifically, if $I$ is piecewise differentiable, then $I' \leq 1$. In this case, retained claims $X - I(X)$ and $X$ are comonotonic risks. Examples of insurance contracts, $I$, that satisfy both restrictions are: (i) deductible coverage, $I(x) = \max(x - d, 0)$ for some $d \geq 0$; (ii) coinsurance, $I(x) = ax$ for some $a \in [0, 1]$; (iii) coverage with a maximum limit, $I(x) = \min(x, u)$ for some $u \geq 0$; or (iv) coverage that combines these three provisions.

Yaari (1987) shows that axiom DU5 is equivalent to the following axiom:

**DU5* If $X, Y$ and $Z$ are pairwise comonotonic, and if $X \prec Y$, then the outcome mixture satisfies**

$$pX + (1 - p)Z \prec pY + (1 - p)Z,$$

**for all $p \in [0, 1]$.**
The equivalence of DU5 and DU5* follows from the fact that if \( X \) and \( Y \) are comonotonic, then the outcome mixture \( (1 - p)X + pY \) has inverse ddf \( (1 - p)S_X^{-1} + pS_Y^{-1} \) (Denneberg, 1994). To put Axiom DU5* in the context of ordering insurance risks, suppose a decision maker (d.m.) – for example, a reinsurer – is asked to choose between the following portfolios of insurance risks: Portfolio 1 consists of risks \( X \) in proportion \( p \) with the remainder of the portfolio made up with risks \( Z \). Portfolio 2 consists of risks \( Y \) in proportion \( p \) with the portfolio made up with risks \( Z \). Also suppose the d.m. views risk \( X \) to be less risky than \( Y \). If risk \( Z \) is a hedge against risk \( Y \), then the d.m. may decide that the hedged Portfolio 2 is less risky than Portfolio 1 even though risk \( Y \) is riskier than \( X \) without the presence of \( Z \). However, if the risks are restricted to be comonotonic, then \( Z \) will not be a hedge against either \( X \) or \( Y \), and by Axiom DU5*, the d.m. will choose Portfolio 1 over Portfolio 2, for the same premium scheme (Yaari, 1987).

Under axioms EU1–4 and DU5 (or DU5*), Yaari (1987) shows that there exists a distortion function \( g \) such that \( X < Y \) iff their certainty equivalents satisfy \( H_g[X] \leq H_g[Y] \), in which one defines a distortion function and certainty equivalents as follows:

**Definition 2.3.** A distortion function \( g \) is a nondecreasing function \( g : [0, 1] \to [0, 1] \) with \( g(0) = 0 \) and \( g(1) = 1 \). For a nonnegative random variable \( X \), the certainty equivalents \( H_g[X] \) and \( H_g[-X] \) are given by

\[
H_g[X] = \int_0^\infty g[S_X(t)] \, dt = \int_0^1 S_X^{-1}(q) \, dg(q),
\]

(2.3a)

and

\[
H_g[-X] = -H_g[X],
\]

(2.3b)

in which \( \tilde{g} \) is the distortion defined by \( \tilde{g}(p) = 1 - g(1 - p) \), \( 0 \leq p \leq 1 \).

The function \( g \) is called a distortion because it distorts the probabilities \( S_X(t) \) before calculating a generalized expected value. Note the duality between expressions (2.2) and (2.3). In (2.2), a utility function changes how wealth is valued, while in (2.3), a distortion changes how tail probabilities are valued. Wang (1996a) discusses pricing insurance risks using \( H_g \) and shows that \( H_g \) satisfies the following properties:

- If \( g(p) = p \) for all \( p \in [0, 1] \), then
  \[
  H_g[X] = E[X].
  \]
- If \( g(p) \geq p \) for all \( p \in [0, 1] \), then
  \[
  H_g[X] \geq E[X].
  \]
- In particular, if \( g \) is a concave distortion, then \( g(p) \geq p \).
  - \( H_g[aX + b] = aH_g[X] + b \) for \( a, b \geq 0 \).
  - For \( X \) and \( Y \) comonotonic,
    \[
    H_g[X + Y] = H_g[X] + H_g[Y].
    \]
- For concave \( g \),
  \[
  H_g[X] \geq E[X] \quad \text{and} \quad H_g[X + Y] \leq H_g[X] + H_g[Y].
  \]
- For convex \( g \),
  \[
  H_g[X] \leq E[X] \quad \text{and} \quad H_g[X + Y] \geq H_g[X] + H_g[Y].
  \]

A distorted probability is a special case of a nonadditive measure; see Denneberg (1994) for more background on nonadditive measure theory.
3. Utility-free ordering of risks

There are two approaches to ordering risks – economic orderings and statistical orderings. Interestingly, both approaches result in the same orderings of risks. This agreement is another important factor that contributed to the popularity of expected utility theory.

3.1. Economic ordering of risks under expected utility theory

One criticism of expected utility theory is that ordering of risks depends on a subjective utility function, unknown to an objective observer, and that different individuals will order risks depending on their utility functions. On the other hand, each individual will totally order the risks, albeit differently. If one is interested in how a collection of decision makers (d.m.’s) orders risks, then the resulting ordering will be a partial ordering.

Within the framework of utility theory, economists define classes of d.m.’s according to characteristics of their utility functions.

Definition 3.1. \( D_n \) is the class of utility functions \( u \) such that \( u \) is at least \( n - 1 \) times differentiable, \((-1)^{k+1}u^{(k)} \geq 0, k = 1, 2, \ldots, n - 1, \) and \((-1)^{n}u^{(n-1)} \) is nonincreasing.\(^1\)

One can use the class \( D_n \) to define a partial ordering on the set of risks. Specifically:

Definition 3.2. Define \( X \prec_n Y \) iff \( E[u(-X)] \geq E[u(-Y)] \), for \( u \in D_n \).

Note that \( X \prec_n Y \) iff \( E[\tilde{u}(X)] \leq E[\tilde{u}(Y)] \), for all utility functions \( \tilde{u} \) such that \( \tilde{u}^{(k)} \geq 0, k = 1, 2, \ldots, n - 1, \) and \( \tilde{u}^{(n-1)} \) is nondecreasing. Indeed, this follows from Definition 3.2 by setting \( \tilde{u}(w) = -u(-w) \). Also, note that \( D_{n+1} \subset D_n \); therefore, if \( X \prec_n Y \), then \( X \prec_{n+1} Y \). We, thus, say that as \( n \) increases, the ordering becomes weaker.

Consider the following three special cases of \( n \):

1. \( n = 1 \): \( D_1 \) is the class of utility functions of d.m.’s who prefer more wealth to less. \( X \preceq_1 Y \) holds iff \( S_X(t) \leq S_Y(t) \) for all \( t \geq 0 \). See, for example, Quiggin (1993, p. 21).

2. \( n = 2 \): \( D_2 \) is the class of utility functions of d.m.’s who are also risk-averse. \( X \prec_2 Y \) holds iff \( \int_1^\infty S_X(t) \, dt \leq \int_1^\infty S_Y(t) \, dt \) for all \( t \geq 0 \) (Quiggin, 1993, pp. 21f).

3. \( n = 3 \): \( D_3 \) contains the utility functions of risk-averse d.m.’s who display decreasing absolute risk aversion, \( D_{\text{DARA}} \). Loosely speaking, the maximum a d.m. in \( D_{\text{DARA}} \) is willing to pay to insure a given risk decreases as her or his wealth increases (Pratt, 1964; Whitmore, 1970).

Note that from item 1 above, the common ordering by d.m.’s who prefer more to less is equivalent to ordering according to the tail probabilities of the distributions. Similarly, from item 2, we see that the common ordering by risk-averse d.m.’s is equivalent to ordering according to the stop-loss premiums of the distributions, the integrated tail probabilities. This equivalence also holds for higher orderings, a well-known result that we state in the next section.

3.2. Stochastic dominance

Statistical orderings of distributions are based on tail probabilities. Denote \( ^1S_X(t) = S_X(t) = \Pr(X > t) \), and define repeated integration of the tail probabilities by

\[
^{n+1}S_X(t) = \int_t^\infty nS_X(u) \, du \quad \text{for } n = 1, 2, \ldots, \text{ and } t \geq 0.
\]

\(^1\)We also include the utility functions \( u_d \) in \( D_n, d \geq 0 \), in which \( u_d \) is defined by \( u_d(w) = -((w - d)_+)^{n-1} \). Note that \( u_d \) has piecewise continuous \((n - 1)\)th derivative.
**Definition 3.3.** If $E[X^k] \leq E[Y^k]$ for $k = 1, 2, \ldots, n - 1$, and if $^nS_X(t) \leq ^nS_Y(t)$ for all $t \geq 0$, then $X$ is said to precede $Y$ in $n$th stochastic order. $Y$ is said to dominate $X$; therefore, this ordering is also referred to as $n$th stochastic dominance.

The following theorem states that $n$th stochastic ordering is equivalent to the ordering by $D_n$; see, for example, Kaas et al. (1994).

**Theorem 3.4.** Let $X$ and $Y$ be loss random variables. $E[u(-X)] \geq E[u(-Y)]$, for all $u \in D_n$, iff $X$ precedes $Y$ in $n$th stochastic order.

Thus, we see that both the economic and statistical approaches to ordering of risks lead to the same utility-free partial orderings. If we let $n$ go to infinity in Theorem 3.4, then we have that $X \prec_{\infty} Y$ iff the moments of $X$ and $Y$ are ordered. This ordering is also equivalent to ordering by infinitely differentiable utility functions with derivatives that alternate in sign. Such utility functions $u$ include the exponential utility functions, $u(w) = 1 - e^{-\alpha w}$, $\alpha > 0$; and the power utility functions $u(w) = wc$, $0 < c < 1$.

We next state a proposition from Kaas et al. (1994) that gives sufficient conditions for two risks to be ordered by $n$th stochastic dominance. First, we define the notion of one ddf surpassing another after crossing a given number of times.

**Definition 3.5.** Suppose that for two risks $X$ and $Y$ there is a partition of $[0, \infty]$ into $n$ consecutive intervals (including possibly degenerate intervals), $I_1, I_2, \ldots, I_n$, such that

$(-1)^{n-k}[S_Y(t) - S_X(t)] \geq 0$ on $I_k$, $k = 1, \ldots, n$,

with strict inequality for at least some $t \in I_k$. Then, we say that $S_Y$ surpasses $S_X$ after crossing $n - 1$ times.

**Proposition 3.6** (Crossing Condition). If $S_Y$ surpasses $S_X$ after crossing $n - 1$ times, and if the first $n - 1$ moments of $X$ and $Y$ are equal, $E[X^k] = E[Y^k]$, $k = 1, 2, \ldots, n - 1$, then $X \prec_n Y$.

If $S_Y$ surpasses $S_X$ after crossing once, with $E[X] \leq E[Y]$, then $Y$ is said to be more dangerous than $X$. We end this section by recalling a result of Müller (1996) in which he shows that two risks $X$ and $Y$ are ordered by second stochastic dominance, $X \prec_2 Y$, iff there exists a sequence of risks $X = X_1, X_2, \ldots$, such that $X_{n+1}$ is more dangerous than $X_n$, $X_n$ converges to $Y$ in distribution, and $E[X_n]$ converges to $E[Y]$. In the next section, we parallel the above development by defining distortion-free orderings of risks.

4. Distortion-free ordering of risks

Parallel to Section 3, there are two approaches to ordering of risks in Yaari’s dual theory – economic orderings and statistical orderings. As in expected utility theory, both approaches result in the same dual orderings of risks.

4.1. Economic ordering of risks under Yaari’s dual theory

One possible criticism of Yaari’s dual theory, as of expected utility theory, is that ordering of risks depends on a subjective distortion function, unknown to an objective observer, and that different individuals will order risks differently depending on their distortion functions. As in expected utility theory, if one is interested in how a collection of decision makers (d.m.’s) orders risks, then the resulting ordering will be a partial ordering, dual to stochastic ordering.

Within the framework of Yaari’s dual theory, one may define classes of d.m. ’s according to characteristics of their distortion functions.
Definition 4.1. $D_n^*$ is the class of distortion functions $g$ such $g$ is at least $n - 1$ times differentiable, $(-1)^{k+1}g^{(k)} \geq 0$, $k = 1, 2, \ldots, n - 1$, and $(-1)^ng^{n-1}$ is nonincreasing. 2

One can use the class $D_n^*$ to define a partial ordering on the set of risks, dual to the partial ordering given by $n$th stochastic dominance. Specifically:

Definition 4.2. Define $X \prec_n^* Y$ iff $H_g[X] \leq H_g[Y]$ for all $g \in D_n^*$.

Note that $X \prec_n^* Y$ iff $H_g[-X] \geq H_g[-Y]$ for all distortions $\tilde{g}$ such that $\tilde{g}^{(k)} \geq 0$, $k = 1, 2, \ldots, n - 1$, and $\tilde{g}^{(n-1)}$ nondecreasing. Indeed, this follows from Definition 4.2 by setting $g(p) = 1 - g(1 - p)$. Also, note that $D_{n+1}^* \subset D_n^*$, therefore, if $X \prec_n^* Y$, then $X \prec_{n+1}^* Y$. We, thus, say that as $n$ increases the dual ordering given in Definition 4.2 becomes weaker.

Consider the following three special cases of $n$:

1. $n = 1$: $D_1^*$ is the class of distortions of d.m.’s who believe that risk increases with increasing probability of loss for any given loss amount. For bounded risks, Yaari (1987) shows that $X \prec_1^* Y$ iff $X \prec_1 Y$, and we obtain this as a corollary to a more general result below.

2. $n = 2$: $D_2^*$ is the class of concave distortions. For bounded risks, Yaari (1987) shows that $X \prec_2^* Y$ iff $X \prec_2 Y$, and we also obtain this as a corollary to a more general result below. For another interpretation of $D_2^*$, consider a Bernoulli risk $X$ that takes on the values $\$0$ and $\$1/p$ with probabilities $1 - p$ and $p$, respectively. The certainty equivalent of $X$ is simply $g(p)/p$, and the risk premium is $g(p)/p - E[X] = g(p)/p - 1$. The risk premium relative to the expected loss, $r(p)$, is also $g(p)/p - 1$.3 The relative risk premium $r(p)$ is increasing with decreasing $p$ if $g$ is concave. So, $D_2^*$ is the class of distortions of d.m.’s who are risk-averse in the sense that their relative risk premia are greater for rare events than for less rare events.

3. $n = 3$: $D_3^*$ is the class of concave distortions that also have increasing second derivatives; i.e., the absolute value of the second derivative of $g$, $|g''|$, increase as $p$ decreases. The relative risk premium for a Bernoulli risk, $r(p)$, introduced in the previous item, is increasing with decreasing $p$ and convex if the distortion $g$ is in $D_3^*$. One might say that a d.m. with distortion in $D_3^*$ is even more risk-averse than one with a distortion in $D_2^*$ (but not $D_2^*$) because the increase in the relative risk premium accelerates as the risk becomes more rare. Ordering of risks by distortions in $D_3^*$ is different from third stochastic dominance, and we explain why this occurs below.

In the next section, we consider the statistical ordering of risks obtained from ordering iterated integrations of the inverse ddf.

4.2. Dual stochastic dominance

As we show below, the dual ordering introduced in Section 4.1 is related to ordering of distributions based on iterated integration of the inverse ddf, $S_X^{-1}$, as given in (2.1). This is not surprising because of the origin of Yaari’s dual theory of risk; recall that he replaces $S_X$ with its inverse in the axioms of expected utility theory. Denote $S_X^{-1}(q) = S_X^{-1}(q), 0 \leq q \leq 1$, and define repeated integration of the inverse ddf by

$$n+1S_X^{-1}(q) = \int_0^q nS_X^{-1}(p) \, dp, \quad \text{for } n = 1, 2, \ldots \quad \text{and } 0 \leq q \leq 1.$$

Note that $nS_X^{-1}$ is not equal to $(nS_X)^{-1}$.

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2 We also include the distortion functions $g_q$ in $D_q^*, 0 \leq q \leq 1$, in which $g_q$ is defined by $g_q(p) = 1 - (q - p)^{n-1}/q^n$. Note that $g_q$ has piecewise continuous $(n - 1)$th derivative.

3 Note that $r(p) = g(p)/p - 1$, regardless the size of the Bernoulli loss.
Definition 4.3. Let $g_k$ denote the distortion defined by $g_k(p) = 1 - (1 - p)^k$, $k$ a positive integer. Define the $k$th dual moment of $X$ by

$$H_k[X] \equiv H_{g_k}[X] = \int_0^\infty [1 - (1 - S_X(t))^k] \, dt.$$ 

If the first $n - 1$ dual moments of $X$ and $Y$ are ordered, i.e., $H_k[X] \leq H_k[Y]$ for $k = 1, 2, \ldots, n - 1$, and if $n S_X^{-1} \leq n S_Y^{-1}$, then we say that $X$ precedes $Y$ in $n$th dual stochastic order. We say that $Y$ dominates $X$, and thus, also refer to this ordering as $n$th dual stochastic dominance.

We have the following theorem, parallel to Theorem 3.4, that shows that $n$th dual stochastic ordering is equivalent to the ordering given by $D^n_n$.

Theorem 4.4. $H_g[X] \leq H_g[Y]$ for all $g \in D^n_n$ iff $X$ precedes $Y$ in $n$th dual stochastic order.

To prove Theorem 4.4, we use the following lemma.

Lemma 4.5. \[ \int_0^\infty [q^{n-1} - (q - S_X(t))^{n-1}] \, dt = (n-1)! S_X^{-1}(q), \] for $n = 1, 2, \ldots$, and $0 \leq q \leq 1$.

Proof. We use induction on $n$ to prove this lemma. First observe that for $n = 1$, the left-hand side equals

$$\int_0^S X^{-1}(q) \, dt + \int_0^\infty 0 \, dt = S_X^{-1}(q) = 0! S_X^{-1}(q),$$

the right-hand side.

Next observe that for $n = 2$, the left-hand side equals

$$\int_0^S X^{-1}(q) \, dt + \int_0^\infty S_X(t) \, dt = \int_0^\infty S_X^{-1}(p) \, dp = 1! S_X^{-1}(q),$$

the right-hand side.

Assume that the result holds for $n = k \geq 2$, and show the result holds for $n = k + 1$. For $n = k + 1$, the right-hand side equals

$$k! S_X^{-1}(q) = k(k - 1)! \int_0^q S_X^{-1}(p) \, dp$$

$$= k \int_0^q \int_0^\infty [p^{k-1} - (p - S_X(t))^{k-1}] \, dt \, dp$$

$$= k \int_0^\infty \int_0^q [p^{k-1} - (p - S_X(t))^{k-1}] \, dp \, dt$$

$$= \int_0^\infty \int_0^q [q^k - (q - S_X(t))^k] \, dt,$$

the left-hand side. Thus, the result holds for $n = 1, 2, \ldots$ \( \square \)

4 Because one is ordering risks according to the inverse dfd and their repeated integrations this ordering is also known as inverse stochastic dominance (Muliere and Scarsini, 1989).
As a special case of this lemma, let $q = 1$ and $n = k + 1$, then the $k$th dual moment $H_k[X]$ of $X$ equals $k^{k+1}S^{-1}_X(1)$. Also, note that $H_k[X] = E[\max(X_1, \ldots, X_k)]$, in which $X_1, \ldots, X_k$ are independent variables, each distributed as $X$.

**Proof of Theorem 4.4.** ‘Only if’: Let $g_k$, $k = 1, 2, \ldots, n - 1$, be as in Definition 4.3. $g_k$ is in $D_\ast^n$; therefore, $H_k[X] \leq H_k[Y]$ for $k = 1, 2, \ldots, n - 1$. Next, let $g_q$ be the distortion given by $g_q(p) = 1 - (q - p)^{n-1}/q^{n-1}$ for a fixed, but arbitrary, $q$ in $[0, 1]$. Note that $g_q$ is in $D_\ast^n$; therefore, $H_{g_q}[X] \leq H_{g_q}[Y]$ for all $q$ in $[0, 1]$, or equivalently, $S^{-1}_X \leq S^{-1}_Y$, by Lemma 4.5.

‘If’: Let $g$ be a distortion in $D_\ast^n$, and expand $g$ in a Taylor series about $1$:

$$g(q) = g(1) + g'(1)(q - 1) + \frac{g''(1)}{2!}(q - 1)^2 + \cdots + \frac{g^{(n-1)}(1)}{(n-1)!}(q - 1)^{n-1}$$

and

$$= g'(1)[1 - (1 - q)] - \frac{g''(1)}{2!}[1 - (1 - q)^2] + \cdots + (-1)^{n-1}\frac{g^{(n-1)}(1)}{(n-1)!}[1 - (1 - q)^{n-1}]$$

$$+ \left[1 - g'(1) + \frac{g''(1)}{2!} - \cdots + (-1)^{n-1}\frac{g^{(n-1)}(1)}{(n-1)!}\right] + \frac{(-1)^{n-1}}{(n-1)!} \int_1^q (p - q)^{n-1} dg^{(n-1)}(p)$$

Evaluate $H_{g}[X]$ and $H_{g}[Y]$ using this expansion together with Lemma 4.5. Since $(-1)^{k}g^{(k-1)} \geq 0$ for $k \leq n$, it follows that $H_{g}[X] \leq H_{g}[Y]$. Because $g \in D_\ast^n$ is arbitrary, we have $X \prec_{\ast}^{\infty} Y$. \(\square\)

Thus, we see that the duals of both the economic and statistical approaches to ordering of risks lead to the same distortion-free partial orderings. If we let $n$ go to infinity in Theorem 4.4, then we have that $X \prec_{\ast}^\infty Y$ iff the dual moments of $X$ and $Y$ are ordered. This ordering is also equivalent to ordering by infinitely differentiable distortions with derivatives that alternate in sign. Such distortions $g$ include the proportional hazards distortions, $g(q) = q^c$, $0 < c < 1$ (Wang, 1996b). Thus, the proportional hazards ordering is weaker than any $n$th dual stochastic ordering.

In the next two corollaries, we show that ordering by first and second stochastic dominance is equivalent to ordering by first and second dual stochastic dominance, respectively. Yaari (1987) proves these results for bounded risks, but they also hold for risks bounded from below.

**Corollary 4.6.** $X \prec_1 Y$ iff $X \prec_{\ast}^1 Y$. 

Proof. \(X \prec Y \iff S_X(t) \leq S_Y(t), t \geq 0 \iff S_X^{-1}(q) \leq S_Y^{-1}(q), q \in [0, 1] \iff X \prec^*_Y.\)

Corollary 4.7. \(X \prec^*_Y \iff X \prec Y.\)

Proof. 'If': Suppose \(X \prec^*_Y.\) \(\int_0^\infty [1 - (1 - S_X(t))] \, dt = E[X],\) so we have that \(E[X] \leq E[Y].\) Let \(d \in \mathbb{R}^+\) be fixed, but arbitrary, and let \(p = S_X(d).\) \(2S_X^{-1} \leq 2S_Y^{-1}\) implies that

\[
pS_X^{-1}(p) + \int_{S_X(p)}^{S_Y(p)} S_X(t) \, dt \leq pS_Y^{-1}(p) + \int_{S_Y(p)}^{S_X(p)} S_Y(t) \, dt,
\]

\[
\Rightarrow \int_{S_X(p)}^{S_Y(p)} S_X(t) \, dt + pd \leq \int_{S_Y(p)}^{S_X(p)} S_Y(t) \, dt + dS_Y^{-1}(p),
\]

because \(\int_{d}^{1} S_X(t) \, dt = (d - S_X^{-1}(p))p.\) It follows that

\[
\int_{d}^{1} S_X(t) \, dt \leq \int_{d}^{1} S_Y(t) \, dt + \int_{S_Y(p)}^{S_X(p)} [S_Y(t) - p] \, dt \leq \int_{d}^{1} S_Y(t) \, dt
\]

because \(\int_{S_Y(p)}^{S_X(p)} [S_Y(t) - p] \, dt \leq 0.\) Thus, \(X \prec^*_Y\) implies that \(X \prec Y.\)

'Only if': From Müller's (1996) work, cited at the end of Section 3.2, it follows that it is enough to show that \(X \prec^*_Y\) for the case in which \(E[X] = E[Y]\) and \(S_Y\) surpasses \(S_X\) after crossing once, say at \(t = c.\) For \(q \leq S_X(c),\) we clearly have \(\int_{0}^{q} S_X^{-1}(p) \, dp \leq \int_{0}^{q} S_Y^{-1}(p) \, dp.\) For \(q > S_X(c),\) \(\int_{q}^{1} S_X^{-1}(p) \, dp \geq \int_{q}^{1} S_Y^{-1}(p) \, dp\) which implies that \(\int_{0}^{q} S_X^{-1}(p) \, dp \leq \int_{0}^{q} S_Y^{-1}(p) \, dp\) because \(E[X] = \int_{0}^{1} S_X^{-1}(p) \, dp,\) and \(E[X] = E[Y].\)

Alternatively, one can prove Corollary 4.7 by noting that if one replaces ddf's in the ordering of dangerousness with inverse ddf's, then the ordering remains unchanged. Also, \(E[X] = \int_{0}^{\infty} S_X(t) \, dt = \int_{0}^{1} S_X^{-1}(p) \, dp.\) Thus, the transitive closure of the ordering of dangerousness, which is equivalent to second stochastic dominance (Müller, 1996), is also equivalent to dual second stochastic dominance.

One might expect that the class of distortion functions \(D_1^*\) would order risks according to the third stochastic ordering because of the duality in the two axiomatic systems. However, we have the following example which shows that distortions in \(D_1^*\) do not necessarily order risks according to the third stochastic ordering.

Example 4.8. Consider two risks \(X\) and \(Y\) given by

\[
\text{Pr}\{X = x\} = \frac{1}{4}, \quad x = 0, 1, 2, 3;
\]

and

\[
\text{Pr}\{Y = 0\} = \frac{1}{6}, \quad \text{Pr}\{Y = 1\} = \frac{1}{2}, \quad \text{Pr}\{Y = 3\} = \frac{1}{3}.
\]

\(X\) and \(Y\) have the same first two moments: \(E[X] = E[Y] = 1.5,\) and \(E[X^2] = E[Y^2] = 3.5.\) Also, one can verify that \(X \prec Y\) because \(S_Y\) surpasses \(S_X\) after crossing twice. However, for \(g \in D_1^*\) given by \(g(p) = \sqrt{p}, H_g[X] = 2.073 > H_g[Y] = 2.068;\) thus, \(X \prec^*_Y\) does not hold in this case.
The basic reason that $X \prec_n^* Y$ does not hold is that $H_2[X] > H_2[Y]$. Indeed,

$$H_2[X] = \frac{1}{3}(1 - (1 - \frac{3}{4})^2) + \frac{1}{3}(1 - (1 - \frac{1}{2})^2) + \frac{1}{3}(1 - (1 - \frac{1}{2})^2) = \frac{17}{24},$$

while

$$H_2[Y] = \frac{1}{3}(1 - (1 - \frac{5}{6})^2) + \frac{2}{3}(1 - (1 - \frac{1}{2})^2) = \frac{25}{36}.$$

We next prove a proposition dual to Proposition 3.6 that gives sufficient conditions for two risks to be ordered by $n$th dual stochastic dominance.

**Proposition 4.9 (Crossing condition).** If $S_Y$ surpasses $S_X$ after crossing $n - 1$ times, and if $H_k[X] = H_k[Y]$, $k = 1, 2, \ldots, n - 1$, then $X \prec_n^* Y$.

**Proof.** Define $W$ on $[0, 1]$ by $W(q) = S_Y^{-1}(q) - S_X^{-1}(q)$. By repeated differentiation, we obtain

$$W^{(k)}(q) = n - k \quad S_y^{-1}(q) - n - k \quad S_x^{-1}(q)$$

for $k = 0, 1, \ldots, n - 1$. By assumption, $W^{(n-1)} = S_y^{-1} - S_x^{-1}$ has at most $n - 1$ sign changes, $W^{(n-1)}$ is positive near zero, and

$$W^{(k)}(1) = n - k \quad S_y^{-1}(1) - n - k \quad S_x^{-1}(1) = \frac{1}{(n - k - 1)!} [H_{n-k-1}(Y) - H_{n-k-1}(X)] = 0 \quad \text{(by Lemma 4.5)}$$

for $k = 0, 1, \ldots, n - 2$. This implies that $W^{(n-2)}$ has at most $n - 2$ sign changes and $W^{(n-2)}$ is positive near zero. By continuing inductively, we obtain that $W$ has no sign changes and is positive near zero; thus, $W$ is nonnegative on $[0, 1]$. Hence, $X \prec_n^* Y$.

In the next section, we examine applications of dual stochastic dominance in insurance and in the economics of income equality.

5. Application of dual stochastic dominance

Kaas et al. (1994) discuss various properties and applications of ordering of risks in insurance, e.g., preserving an order under adding risks using an order to determine optimal reinsurance. We look at similar issues in the framework of Yaari’s dual theory. In Section 5.1, we show how the third dual stochastic ordering orders three parametric families of distributions. In Section 5.2, we examine which “actuarial operations” preserve the dual ordering. Then, we show how to apply the dual ordering to determine optimal reinsurance. Finally, we study the relationship between Yaari’s dual theory of risk and measuring income inequality, and show how the $k$th dual moments, $H_k[X] = \int_0^\infty [1 - (1 - S_X(t))^k] dt$, $k = 1, 2, \ldots$, are related to the Gini index, a relative measure of income inequality.

5.1. Ordering families of distributions

We equate the first two dual moments, $H_1$ and $H_2$, among three families of distributions, and determine how the families are related with respect to third dual stochastic ordering. Let $X$ be a discrete Bernoulli random variable with values at 0 and $x > 0$ with probabilities $1 - p$ and $p$, respectively. Let $Y$ be distributed according to the exponential-inverse gaussian $(\mu, \beta)$ distribution with ddf $S_Y(t) = \exp[(\mu/\beta)(1 - \sqrt{1 + 2\beta t})]$, $t \geq 0, \mu > 0, \beta > 0$. Finally, let $Z$ be distributed according to the Pareto $(\alpha, \lambda)$ with ddf $S_X(t) = \lambda/((\lambda + t)\alpha)$, $t \geq 0, \alpha > 1, \lambda > 0$ (Hesselager, et al., 1997).
When we equate the first two dual moments, we obtain the relationships $\alpha = 3/2 + \mu/\beta$, $\lambda = (\beta + \mu)/(2\beta + 2\mu)$, $p = (\beta + 2\mu)/(4(\beta + \mu))$, and $x = 4(\beta + \mu)^2/(\beta + 2\mu)$. In this case, the three random variables are ordered $X <_n^* Y <_n^* Z$. Indeed, $3S_X^{-1} \leq 3S_Y^{-1} \leq 3S_Z^{-1}$, but we omit the details.

5.2. Ordering preserving operations

The partial ordering defined by $n$th dual stochastic dominance is preserved under summing risks, as stated more precisely in the following proposition.

**Proposition 5.1.** If $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_m$ are sequences of risks with $X_i <_n^* Y_i$, $i = 1, 2, \ldots, m$, and with $Y_1, Y_2, \ldots, Y_m$ pairwise comonotonic, then $\sum_{i=1}^m X_i <_n^* \sum_{i=1}^m Y_i$.

To prove this proposition, we use the following lemma due to Wang and Dhaene (1997).

**Lemma 5.2.** Let $X$ and $Y$ be risks, and let $U$ be any uniformly distributed random variable on $[0, 1]$, then $X + Y <_n^* S_X^{-1}(U) + S_Y^{-1}(U)$.

**Proof of Proposition 4.9.** We prove the proposition by induction on $m$. The result is trivially true when $m = 1$. Suppose the result is true for $m = k$, and show that it is true for $m = k + 1$.

$$
\sum_{i=1}^{k+1} X_i = \left( \sum_{i=1}^k X_i \right) + X_{k+1} <_n^* \frac{S_{X_{k+1}}^{-1}}{\sum_{i=1}^k X_i} (U) + \frac{S_X^{-1}}{\sum_{i=1}^k X_i} (U) <_n^* \frac{S_{Y_{k+1}}^{-1}}{\sum_{i=1}^k Y_i} (U) + \frac{S_Y^{-1}}{\sum_{i=1}^k Y_i} (U) <_n^* \sum_{i=1}^{k+1} Y_i,
$$

in which $X <_n^* Y$ means that both $X <_n^* Y$ and $Y <_n^* X$ hold. \(\square\)

Dhaene et al. (1997) also prove Proposition 5.1 for $n = 2$ and use it to obtain maximal aggregate stop-loss net premiums.

While $n$th stochastic dominance is preserved by the largest-order statistics among independent risks (van Heerwaarden, 1991), $n$th dual stochastic dominance is not preserved by the largest-order statistics among comonotonic risks. For example, let $X$ be uniformly distributed on $[0, 1]$. Let $Y$ be a random variable with ddf $S_Y(t) = 1 - 2t$ for $0 \leq t < 1/4$, $1/4$, for $1/4 \leq t < 1$; and $0$ for $t \geq 1$. Let $Z$ be a Bernoulli $(1/4)$ random variable. Note that $X <_2^* Y$. Suppose $X, Y$ and $Z$ are comonotonic; then, in this example, $S_{\max(X,Z)} = \max(S_X, S_Z) \geq \max(S_Y, S_Z) = S_{\max(Y,Z)}$, from which it follows that $\max(Y, Z) <_1^* \max(X, Z)$. Similar examples exist to show that $n$th dual stochastic dominance is not preserved by the largest-order statistics among independent risks nor by the smallest order statistics for $n \geq 3$.

Because of the axiom EU5, $n$th stochastic dominance is preserved by probabilistic mixtures of risks but not necessarily by outcome mixtures of risks, unless the risks are independent. Similarly, because of axiom DU5$, $n$th dual stochastic dominance is preserved by outcome mixtures of comonotonic risks but not necessarily by probabilistic mixtures. This result is stated more precisely in the following corollary of Proposition 5.1.

**Corollary 5.3.** Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_m$ be sequences of risks with $X_i <_n^* Y_i$, $i = 1, 2, \ldots, m$, and with $Y_1, Y_2, \ldots, Y_m$ pairwise comonotonic. Let $p_i, i = 1, 2, \ldots, m$, be nonnegative weights summing to 1, then $\sum_{i=1}^m p_i X_i <_n^* \sum_{i=1}^m p_i Y_i.$
5.3. Optimal reinsurance

Suppose we use the valuation operator defined by $H_g[X] = \int_0^\infty g[S_X(t)] \, dt$ for a given concave distortion $g$. The crucial feature of this valuation operator is that it preserves second (dual) stochastic dominance. Denote the feasible set of reinsurance contracts by $I_P = \{ I : I(0) = 0, 0 \leq I' \leq 1, E[I(X)] = P \}$. Following van Heerwaarden (1991), we define the optimal reinsurance contract for a given budget constraint $P$ as follows.

Definition 5.4. The reinsurance contract $I^* \in I_P$ is the optimal reinsurance contract with respect to $H_g$ if $H_g[X - I^*(X)] < H_g[X - I(X)]$ for all $I \in I_P$. That is, the valuation operator applied to the retained claims, $X - I(X)$, is minimum for the optimal reinsurance contract.

The following proposition shows that the optimal reinsurance contract is stop-loss reinsurance with a deductible determined by the budget constraint $P$, independent of the given concave distortion $g$. Thus, stop-loss insurance is optimal for all d.m.’s in $D^*_2$. This proposition is a special case of Theorem 9.3.2 of van Heerwaarden (1991), but we include the proof because it is rather interesting.

Proposition 5.5. The optimal reinsurance contract $I^* \in I_P$ is of the form $I^*(x) = (x - d)^+$, in which $d$ is defined by $E[(X - d)^+] = P$ and in which $g$ is a concave distortion.

Proof. We wish to minimize $H_g[X - I(X)] = \int_0^\infty g[S_{X-I}(t)] \, dt = \int_0^1 S_{X-I}^{-1}(q) \, dg(q)$ subject to $E[I(X)] = P$. Note that, for a concave $g$, the measure given by $dg(q)$ is largest for small $q$. Therefore, to minimize $\int_0^1 S_{X-I}^{-1}(q) \, dq$, with the constraint that $\int_0^1 S_{X-I}^{-1}(q) \, dq = E[X] - P$, let $S_{X-I}^{-1}(q)$ be as small as possible (namely, $S_X^{-1}$) for $q$ close to 1. Thus, the inverse ddf of the optimal retained claims is $S_{X-I}^{-1}(q) = \min[d, S_X^{-1}(q)]$, from which it follows that the optimal reinsurance is given by $I^*(x) = (x - d)^+$. □

We also obtain a corollary to Proposition 5.5 which is similar to Theorem 9.4.1 of van Heerwaarden (1991) if the reinsurance contract is based on the claims of each contract in a pool of reinsured contracts. Our result differs from the one in van Heerwaarden in that she considers independent risks, and we consider comonotonic ones.

Corollary 5.6. Let $X_1 + X_2 + \cdots + X_m$ be a sum of $m$ comonotonic risks, each distributed as $X$, and suppose the reinsurance contract $T$ is of the form $T(m, x_1, \ldots, x_m) = \sum_{j=1}^m I(x_j)$, with $I \in I_P$. Then the optimal reinsurance contract $T^*$ is of the form $T^*(m, x_1, \ldots, x_m) = \sum_{j=1}^m (x_j - d)^+$, with $E[(X - d)^+] = P$.

Proof. For each claim $X_j$, the retained part under the deductible $d$ is smaller in $H_g$ ordering than any other feasible retained risk. By Proposition 5.1, $H_g$ is preserved under adding comonotonic risks from which the result follows. □

Utility theory and Yaari’s dual theory define two kinds of risk-aversion: (1) Utility theory expresses risk-aversion as an attitude toward wealth, while it is linear with respect to probability. (2) Yaari’s dual theory expresses risk-aversion as an attitude toward probability (uncertainty), while it is linear with respect to wealth. It may be interesting to investigate the interaction of two theories. For example, suppose that the primary company has a capacity constraint, and thus, risk-aversion is modeled by a utility function of wealth. On the other hand, suppose the reinsurer has sufficient capacity but is mainly concerned with the profitability in the face of uncertainty; thus, the risk-aversion of the reinsurer can be modeled by a distortion function. We suggest that future research examine the interaction of these two kinds of risk-aversion in determining optimal reinsurance.
5.4. Dual moments and the Gini index

To measure income inequality, economists have developed the Gini index.\(^5\) Let \(S_X(t) = \Pr\{X > t\}\) be the proportion of individuals in a society with incomes greater than \(t \geq 0\). The (relative) Gini index of income inequality of \(X\), \(G[X]\), is defined by

\[
G[X] = \frac{E|X_1 - X_2|}{2E[X]},
\]

in which \(X_1\) and \(X_2\) are independent variables, each equal to \(X\) in distribution.

Equivalent expressions for \(G[X]\) are

\[
G[X] = 1 - \int_0^\infty S_X(t)^2 \, dt = \frac{E[\max(X_1, X_2)]}{E[X]} - 1.
\]

These expressions for \(G[X]\) lead directly to the relationships

\[
H_2[X] = E[X](1 + G[X]) = E[\max(X_1, X_2)].
\]

Therefore, among risks with equal expected values of the maximum order statistic between \(X_1\) and \(X_2\); thus, it is a measure of the right skewness of \(X\).

One property of the Gini index is that if a sufficiently small amount of money is transferred from a person with income greater than the average to someone with income less than the average, then \(G\) decreases, or equivalently, income inequality decreases. This property is simply a restatement of the fact that \(G\) preserves the ordering in dangerousness among risks with equal means. In fact, this property is preserved more generally by any (absolute) measure of income inequality of the form

\[
H_g[X] = \int_0^\infty g[S_X(t)] \, dt,
\]

with \(g\) a concave distortion, by Yaari (1988) and by our Corollary 4.7. Thus, the \(k\)th dual moment, \(H_k[X] = \int_0^\infty [1 - (1 - S_X(t))^k] \, dt\), \(k = 2, 3, \ldots\), can serve as an absolute measure of income inequality. Note that \(H_1[X] = E[X]\).

Because \(H_k\), \(k = 2, 3, \ldots\), is of the form in expression (2.3) with concave distortion, it satisfies the dual axioms and the properties listed in (2.4). Of special interest are the properties of scale and translation preserving (2.4c), comonotonic additivity (2.4d), and subadditivity (2.4e). Some insightful formulas for \(H_k\), \(k = 1, 2, \ldots\), are given in the following proposition.

**Proposition 5.7.** Let \(X_1, X_2, \ldots, X_k\), be independent variables, \(k = 1, 2, \ldots\), each distributed as \(X\). Denote the \(l\)th-order statistic of \(X_1, X_2, \ldots, X_k\) by \(X_{(l)}\). Then

\[
H_k[X] = E[X_{(k)}] = E[X] \sum_{l=1}^{k-1} \frac{1}{k} E[(X_{(k)} - X_{(l)})].
\]

Thus, \(H_k\) is a measure of the excess of \(X_{(k)}\) over the remaining \(X_{(l)}, l = 1, 2, \ldots, k - 1\).

\(^5\) Recent references include Yaari, 1988; Chakravarty, 1988; Muliere and Scarsini, 1989; Ben Porath and Gilboa, 1994; Weymark, 1995.
Proof. The relationship $S_{X(k)} - 1 - (1 - S_X)^k$ implies the first expression for $H_k$. The second follows from

$$S_X + \frac{1}{k} \sum_{i=1}^{k-1} (S_{X(i)} - S_{X(i)}) = S_{X(k)} + \frac{1}{k} \sum_{i=1}^{k} (S_X - S_{X(i)}) = S_{X(k)}. \quad \square$$

In the special case of a continuous random variable $X$, we have the following proposition.

**Proposition 5.8.** Let $X$ be a continuous random variable. Denote the cumulative distribution function of $X$ by $F_X(t) = \Pr\{X \leq t\}, t \geq 0$. Then,

$$H_k[X] = E[X] + k \text{Cov}(X, F_X(X)^{k-1}),$$

for $k = 1, 2, \ldots$

**Proof.** The result follows from integration by parts.

$$k \text{Cov}(X, F_X(X)^{k-1}) = k \int_0^\infty (t - E[X]) F_X(t)^{k-1} dF_X(t)$$

$$= -(t - E[X])(1 - F_X(t)^{k}) \bigg|_0^\infty + \int_0^\infty (1 - F_X(t)^{k}) dt$$

$$= -E[X] + H_k[X]. \quad \square$$

The expression for $H_k$ in Proposition 5.8 does not hold in general. For example, consider the Bernoulli random variable $X = 0, 1$, with probabilities $\frac{3}{4}, \frac{1}{4}$, respectively. Then $H_2[X] = \frac{7}{16}$, but $E[X] + 2 \text{Cov}(X, F_X(X)) = \frac{11}{12}$.

As a suggestion for further application of dual stochastic dominance, we encourage the interested reader to examine the significance of using dual third stochastic dominance to order income distributions. Davies and Hoy (1994) examine the normative implications of using third stochastic dominance to order income distributions. They show that a sufficiently small transfer of wealth from a rich person to a richer person can be offset by a transfer of wealth from a rich person to a poor person so that the combined transfer results in a more equitable income distribution. This result is essentially due to the twice-crossing condition as stated in Proposition 3.6 with $n = 3$.

If one were to follow Yaari’s dual theory (1988) in the context of income inequality, one would replace stochastic dominance with dual stochastic dominance. It would be natural to apply Proposition 4.9 with $n = 3$ and obtain results parallel to those of Davies and Hoy (1994) in Yaari’s framework. By so doing, one could examine the robustness of the results of Davies and Hoy in a theory dual to utility theory. Also, Yaari’s dual theory may prove to be a more natural setting for considering income inequality, as demonstrated above in relating $H_2$ to $G$ and as shown in Yaari (1988).

### 6. Summary

Under Yaari’s theory we developed the dual to $n$th stochastic ordering and showed that the $n$th dual stochastic ordering is equivalent to the partial ordering determined by a class of distortion functions. We believe that these dual orderings will be useful in testing the robustness of results obtained by considering stochastic dominance orderings, which are essentially rooted in utility theory. We gave an alternative proof to the fact that second stochastic dominance and its dual are equivalent and showed that, for higher orderings, this equivalence does not hold. Therefore, in general, the $n$th dual stochastic ordering is different from stochastic dominance. We gave a motivation from insurance
economics for considering the third dual stochastic dominance and provided applications in insurance and in income inequality.

By researching parallel theories, one often gains new insight into the original theory and a framework in which to examine the robustness of results obtained under the original theory. We hope this work helps broaden the understanding of both utility theory and Yaari's dual theory of risk.

Acknowledgements

We thank Ole Hesselager for every helpful comments, especially for improving our proof of Proposition 4.9. We also thank an anonymous referee for helping us to improve our paper.

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