Abstract

The author recently proposed a premium calculation principle based on proportional hazards (PH) transforms. It is shown that this premium principle resembles the risk-neutral valuation in financial economics, but differs from the traditional utility theory approach. The PH-transform does preserve the stop-loss order of risks which is shared by all risk-avers with increasing concave utility functions.

Keywords: Premium calculation; PH-transform; Ordering of risks

1. Introduction

Risk loadings are required by insurers as a source of solvency margin and potential profit. How to decide on risk loading according to different risk characteristics is a main task in insurance pricing.

An insurance risk $X$ is a non-negative random variable with distribution function $F_X(t) = \Pr\{X \leq t\}$ or survivor function $S_X(t) = 1 - F_X(t)$, which quantifies the size and possibility of potential losses associated with an insurance contract. For notational convenience, we shall not distinguish between a risk $X$ and its loss distribution $F_X$. A premium principle is a rule to assign a premium value to a given risk

$$\pi : F_X \mapsto [0, \infty).$$

Traditional premium principles are mainly based on the first two moments or utility theory (see Goovaerts et al., 1984). It is well known that the first two moments cannot rightly reflect the level of insurance risk since loss distributions are often highly skewed. Utility theory provides a conceptual framework for insurance premium calculation, but it fails to be used commonly in practice because of the difficulties associated with implementation.

Recently actuaries have become more aware of the resemblance of insurance risks and financial risks. While risk-neutral valuation has become a standard methodology in pricing financial risks, actuaries are far from agreeing upon one sound premium principle.

In a recent paper, the author (Wang, 1995) proposed a new premium principle based on proportional hazards (PH) transforms, which can be summarized as follows.

Definition 1. The proportional hazards (PH) transform is a mapping

$$\Pi_\rho : X \mapsto Y$$

such that

$$S_Y(t) = S_X(t)^{\frac{1}{\rho}} \quad (\rho > 0).$$  

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Definition 2. For a risk $X$, the risk-adjusted premium is the mean of the transformed distribution $Y = \Pi_\rho (X)$:

$$\pi_\rho (X) = E[\Pi_\rho (X)] = \int_0^\infty S_X(t)^{\frac{1}{\rho}} \, dt, \quad \rho \geq 1, \quad (2)$$

where $\rho \geq 1$ is called the (risk-averse) index.

The PH-transform principle has many desirable properties, including: (i) scale-invariance $\pi_\rho (aX) = a\pi_\rho (X)$, and (ii) translation-invariance $\pi_\rho (X + b) = \pi_\rho (X) + b$. It also produces additive premiums when pricing excess-of-loss layers. An excess-of-loss layer $(a, b]$ of a risk $X$ is defined by

$$I_{[a,b]} = \begin{cases} 
0, & 0 \leq X < a; \\
(X - a), & a \leq X < b; \\
(b - a), & b < X.
\end{cases}$$

When a risk $X$ is divided into layers $\{(x_i, x_{i+1}) \mid i = 0, 1, \ldots\}$:

$$X = I_{[0,x_1]} + I_{[x_1,x_2]} + \cdots, \quad 0 = x_0 < x_1 < x_2 < \cdots$$

the layer premiums are additive,

$$\pi_\rho (X) = \sum_{i=0}^{\infty} \pi_\rho (I_{[x_i,x_{i+1}]}).$$

In this paper, it is shown that this PH-transform principle resembles the risk-neutral valuation method in option pricing theory, but differs from the traditional utility theory in the economics of insurance. Traditional utility theory yields a natural ordering of risks shared by all risk-avers. It is shown that the PH-transform does preserve this ordering of risks.

2. Risk-neutral valuation

The pricing of insurance risks resembles the pricing of financial risks in: (i) the risk-aversion of investors (insurers), and (ii) the risk-reward trade-off relationship. In financial economics, risk-neutral valuation has become a fundamental methodology (Cox and Ross, 1976; Harrison and Kreps, 1979). Gerber and Shiu's (1994) Esscher transform method is such an example which creates pseudoprobability measures under which the investors are risk-neutral.

Mostly inspired by the success of the risk-neutral valuation methodology in finance, many actuaries propose premium calculations by using transformed distributions.

1. In summarizing the basic results in Bühmann (1980), Bühmann (1985, p.96) wrote: “The effect of a competitive market can be viewed as if the basic probabilities were changed. If state $\omega$ has objective probability $p(\omega)$ the market premium will be obtained from the modified probabilities

$$e^{\alpha Z(\omega)} p(\omega)$$

$$\sum_{\omega'} e^{\alpha Z(\omega') p(\omega')}$$

where $Z(\omega)$ stands for the total of claims to the market if state $\omega$ happens and where $\alpha$ is a measure of total risk aversion”.

2. Cited from Delbaen and Haezendonck (1989, pp. 269–270): “Sondermann (1988) proved that if no arbitrage opportunities exist, then premiums are given by mathematical expectation, not with respect to the original probability distribution but with respect to a new probability distribution called the risk-neutral distribution. The risk-neutral probability distribution changes the original probability distribution involved in order to give more weight to unfavorable events in a risk-averse environment. In financial economics this leads to the concepts of ‘price of risk’ and in insurance mathematics it should explain the safety loading.”

3. Venter (1991) discussed the no-arbitrage implications of insurance pricing, especially the effect of layering a risk. He observed that the only premium principles that preserve layer additivity are those that can be generated from transformed distributions, where the premium for any layer is the expected loss for that layer under the transformed distribution.

It is easy to see that under the PH-transform $Y = \Pi_\rho (X)$ and $X$ have the same null-sets.

If $X$ is continuous with density function $f_X(t)$, $t \in I$, then $Y = \Pi_\rho (X)$ is also continuous with density function

$$f_Y(t) = \left[ \frac{1}{\rho} S_X(t)^{\frac{1}{\rho} - 1} \right] f_X(t), \quad t \in I.$$
One can see that the weight function \( S_x(t)^{\frac{1}{2}} \) increases with the loss size \( t \), thus gives more weight to the unfavorable events.

If \( X \) is defined on discrete points \( \{t_0, t_1, \ldots\} \) with probabilities

\[
p_x(t_i) = \Pr(X = t_i), \quad i = 0, 1, 2, \ldots,
\]

then the transformed probability distribution \( Y = \Pi_\rho(X) \) is also defined on \( \{t_0, t_1, \ldots\} \) with probabilities:

\[
p_Y(t_i) = S_x(t_{i-1})^\frac{1}{2} - S_x(t_i)^\frac{1}{2}, \quad i = 1, 2, \ldots,
\]

\[
p_Y(t_0) = 1 - S_x(t_0)^\frac{1}{2}.
\]

To summarize, the PH-transform modifies the original probability of occurrence and creates pseudoprobabilities (as is the jargon in option pricing theory); on the other hand, the support (all possible loss sizes) is unchanged under PH-transforms. The PH-transform method can be viewed as a risk-neutral valuation. In the risk-neutral world, the insurer is only concerned about the expected loss, which gives the risk-adjusted premium \( \pi(\rho, X) = E[\Pi_\rho(X)] \).

3. Utility theory

Traditionally, utility theory plays a dominant role in the economics of insurance (e.g., Borch 1961, 1975). A decision-maker assigns a utility value \( u(w) \) according to the wealth level \( w \). When facing uncertain economic prospects, the decision-making process can be modeled as maximizing one’s expected utility. For example, if one has initial wealth \( w \) and faces a potential loss \( X \), then the maximum premium \( P \) one is willing to pay for insuring risk \( X \) can be determined by

\[
u(w - P) = E[u(w - X)].
\]

Since the initial wealth \( w \) may be hardly known, it is often ignored in the analysis and one gets the zero utility principle:

\[
u(-P) = E[u(-X)].
\]

One can see that a utility function modifies the value of the wealth function while keeping the probability of occurrence unchanged. By contrast, the PH-transform modifies the probability of occurrence while keeping the wealth function unchanged. This is a fundamental distinction between two approaches. While the PH-transform principle is layer-additive, premiums based on utility functions do not satisfy layer-additivity unless the utility function is linear.

Goovaerts et al. (1984) studied a dozen of premium calculation principles, most of which relate to utility theory, namely:

- (0) net premium principle,
- (1) expected value principle,
- (2) maximum loss principle,
- (3) variance principle,
- (4) standard deviation principle,
- (5) exponential principle,
- (6) mean value principle,
- (7) zero utility principle,
- (8) Swiss principle,
- (9) Orlicz principle,
- (10) Esscher principle.

Reich (1986) showed that, except for the net premium principle, none of the above premium principles satisfies both scale-invariance and translation-invariance.

The relatively recent Dutch principle (Van Heerwaarden and Kass, 1992):

\[
\pi : X \mapsto E(X) + \alpha(X - E(X))_+, \quad 0 \leq \alpha \leq 1,
\]

is a remarkable exception since it does satisfy scale-invariance and translation-invariance. It is noted that the Dutch principle is not layer additive. A simple example is to let \( \alpha = 1 \) and \( \Pr(X = 0) = \Pr(X = 1) = \Pr(X = 2) = \frac{1}{3} \). One gets \( \pi(I_{(0,1)}) + \pi(I_{(1,2)}) = 1 + \frac{1}{3} = \pi(X) = 1 \frac{1}{3} \).

From the properties of various traditional premium principles, we can see that the PH–transform method is not equivalent to any of them.

4. The PH-transform order of risks

Any premium calculation principle implicitly yields an order among risks: higher risks are assigned higher premiums. For some risks there is a common ordering shared by all prudent insurers.

Definition 3. A risk \( U \) is smaller than risk \( V \) in PH-transform order (notation \( U \preceq_{\text{PH}} V \)) if \( \pi(\rho)(U) \preceq_{\text{PH}} V \).
\( \pi_\rho(V) \) for all \( \rho \geq 1 \).

If risk \( U \) is smaller than risk \( V \) in PH-transform order, every insurer with index \( \rho \geq 1 \) would assign a higher premium to risk \( V \).

Traditionally, there are some basic orderings of risks which are closely related to utility theory. Here we summarize some of the basic results in this area, for details one can consult Kaas et al. (1994) or Goovaerts et al. (1991). In this section we investigate whether the PH-transform order agrees with the basic ordering of risks in traditional economics of insurance.

**Definition 4.** A risk \( U \) is smaller than risk \( V \) in stochastic order (notation \( U \preceq_{st} V \)) if any of the following two equivalent conditions is met:

1. For every decision-maker with an increasing utility function \( u \):
   \[
   E[u(-X)] \geq E[u(-Y)].
   \]
2. \( S_U(t) \leq S_V(t) \) for all \( t \geq 0 \).

**Definition 5.** A risk \( U \) is less dangerous than risk \( V \) (notation \( U \preceq_{D} V \)) if

1. \( E(U) \leq E(V) \);
2. there exists \( c > 0 \) such that
   \[
   S_U(t) \begin{cases} 
   \geq S_V(t), & 0 \leq t < c; \\
   \leq S_V(t), & c \leq t.
   \end{cases}
   \]

**Theorem 1.** The PH-transform preserves the ordering of dangerousness:

(i) \( U \preceq_{D} V \implies U \preceq_{PH} V \),

(ii) \( U \preceq_{D} V \iff \Pi_\rho(U) \preceq_{D} \Pi_\rho(V) \).

**Proof.** (i) Let \( U \preceq_{D} V \), with \( c \) as in Definition 5. We define

\[
\text{LHS}(\rho) = \int_0^c S_U(t)^{\frac{1}{\rho}} - S_V(t)^{\frac{1}{\rho}} \, dt,
\]

\[
\text{RHS}(\rho) = \int_c^\infty S_V(t)^{\frac{1}{\rho}} - S_U(t)^{\frac{1}{\rho}} \, dt.
\]

From \( E(U) \leq E(V) \) we have \( \text{LHS}(1) \leq \text{RHS}(1) \).

We shall use the fact that, for \( \rho > 1 \), \( y^{\frac{1}{\rho}-1} \) is a decreasing function of \( y \in (0, \infty) \), and \( S_V(t)^{\frac{1}{\rho}-1} \) is an increasing function of \( t \in (0, \infty) \).

For \( t < c \), \( S_U(t)^{\frac{1}{\rho}-1} \leq S_V(t)^{\frac{1}{\rho}-1} \leq S_V(c)^{\frac{1}{\rho}-1} \), so we have

\[
\text{LHS}(\rho) = \int_0^c S_U(t)^{\frac{1}{\rho}-1} [S_U(t) - S_V(t)] \, dt \\
\leq S_V(c)^{\frac{1}{\rho}-1} \text{LHS}(1).
\]

For \( t \geq c \), \( S_U(t)^{\frac{1}{\rho}-1} \leq S_V(t)^{\frac{1}{\rho}-1} \geq S_V(c)^{\frac{1}{\rho}-1} \), so we have

\[
\text{RHS}(\rho) \geq \int_c^\infty S_V(t)^{\frac{1}{\rho}-1} [S_V(t) - S_U(t)] \, dt \\
\geq S_V(c)^{\frac{1}{\rho}-1} \text{RHS}(1).
\]

Therefore, \( \text{LHS}(\rho) \leq \text{RHS}(\rho) \) and \( \pi_\rho(U) \leq \pi_\rho(V) \) for all \( \rho \geq 1 \).

(ii) From the definition of the ordering in dangerousness we have

\[
[S_U(t)]_{\rho} \begin{cases} 
\geq [S_V(t)]_{\rho}, & 0 \leq t < c; \\
\leq [S_V(t)]_{\rho}, & c \leq t.
\end{cases}
\]

Together with (i) we conclude that \( \Pi_\rho(U) \preceq_{D} \Pi_\rho(V) \) for all \( \rho > 1 \). \( \square \)

**Definition 6.** A risk \( U \) is smaller than risk \( V \) in stop-loss order (notation \( U \preceq_{st} V \)) if any of the following four equivalent conditions is met:

1. For every decision-maker who has an increasing concave function \( u \) with \( u'(t) \geq 0 \) and \( u''(t) \leq 0 \):
   \[
   E[u(-X)] \geq E[u(-Y)].
   \]
2. The stop-loss transform satisfies the inequality
   \[
   \int_0^x S_U(t) \, dt \leq \int_0^x S_V(t) \, dt
   \]
   for all \( x \geq 0 \).
3. \( E[V|U] \geq U \) with probability one.
4. There exists a sequence of distribution functions \( \{G_1, G_2, \ldots\} \) such that \( F_U = G_1, G_i \preceq_{D} G_{i+1} \) with \( F_V(x) = \lim_{i \to \infty} G_i(x) \) and \( \lim_{i \to \infty} E(G_i) = E(V) \).

The equivalent condition (4) states the structural relationship between the ordering of dangerousness
and the stop-loss order, which is explained in detail in Kaas and Van Heerwaarden (1992) and Müller (1995).

From the equivalent condition (4) we have the following result.

**Corollary 1.** The PH-transform preserves the stop-loss order of risk:

(i) \( U \preceq_{sl} V \implies U \preceq_{PH} V \),

(ii) \( U \preceq_{sl} V \implies \Pi_{\rho}(U) \preceq_{sl} \Pi_{\rho}(V) \).

**Remark** Even though the PH-transform method is totally different from the traditional utility theory approach, they do share some ‘opinions’ on the ordering of risks. If all risk-avers with a concave utility function think that risk \( U \) is less risky than \( V \), the PH-transform preserves this order and yields a higher premium for \( V \).

In general, the stochastic order is stronger than the order in dangerousness, which is stronger than the stop-loss order, which in turn is stronger than the PH-transform order:

\[ U \preceq_{sl} V \implies U \preceq_{D} V \implies U \preceq_{sl} V \implies U \preceq_{PH} V. \]

5. An example

We now give an example to illustrate that the PH-transform order is more discriminative than the stop-loss order.

**Example 1.** Consider the following two loss distributions:

\[ S_{U}(t) = \begin{cases} 
1, & 0 \leq t < 4, \\
0, & 4 < t. 
\end{cases} \] (two-point)

\[ S_{V}(t) = \left( \frac{1}{1 + t} \right)^2 \] \( (\text{Pareto}), \)

where both have the same mean (= 1). While the two-point risk \( U \) has a finite variance (= 3), the variance for the Pareto risk \( V \) is infinite.

By calculating the stop-loss transforms one can see that there is no stop-loss order between the two-point risk \( U \) and the Pareto risk \( V \). In other words, the stop-loss order cannot distinguish which one is a higher risk.

Nevertheless, most prudent insurers would think the Pareto risk \( V \) as more risky since it has a longer tail with an infinite variance. In fact, a deeper theoretical analysis (Kaas et al., 1994, pp.50-53) shows that all regular decision-makers having a smooth (to be precise, twice differentiable) increasing concave utility function perceives the Pareto risk \( V \) as a higher risk.

The PH-transform yields the following premiums (see Fig. 1):

\[ \pi_{\rho}(U) = 4^{\frac{1}{\rho} - \frac{1}{\rho}} \leq \pi_{\rho}(V) = \begin{cases} 
\frac{\rho}{\rho - 2}, & \rho < 2; \\
\infty, & \rho \geq 2. 
\end{cases} \]

From Fig. 1 we can see that, for any \( \rho > 1 \), the PH-transform always assigns a higher premium to the Pareto risk. In other words, there is a PH-transform order: \( U \preceq_{PH} V \).

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**References**


