Lecture 24

Proposition: Linear combinations of normal rv's are normal.

Corollary: Suppose that $X_1, \ldots, X_n$ is a sample from the $\mathrm{Normal}(\mu, \sigma^2)$ distribution. Then

$$\bar{X} \sim \mathrm{Normal}(\mu, \sigma^2/n)$$

Example: Determine the distribution of the rv $Y = 2X_1 - X_2 + 3X_3 + 3$ **where** X_1**,** X_2 **and** X_3 **are independent,** $X_1 \sim \text{Normal}(4, 3)$**,** $X_2 \sim \text{Normal}(5, 7)$ **and** $X_3 \sim \text{Normal}(6, 4)$**.**

Example: Determine the distribution of the rv $Y = X_1 - X_2$ **where** $\text{Cov}(X_1, X_2) = 6$**,** $X_1 \sim \text{Normal}(5, 10)$ **and** $X_2 \sim \text{Normal}(3, 8)$**.**

You are not responsible for complete understanding of the following example. However, it gives some insight as to why linear combinations of normals are normal.

Example: When X and Y are independent standard normal, then $Z = X + Y \sim \mathrm{Normal}(0, 2)$.

$$
\begin{aligned}
\mathrm{P}(Z \le z) &= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \; dx \; dy \\
&= \int_{y=-\infty}^{\infty} \int_{u=-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-(u-y)^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \; du \; dy \\
&= \int_{u=-\infty}^{z} \int_{y=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2+uy-y^2} \; dy \; du \\
&= \int_{u=-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \int_{y=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(y^2-uy)} \; dy \; du \\
&= \int_{u=-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \int_{y=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(y-u/2)^2+u^2/4} \; dy \; du \\
&= \int_{u=-\infty}^{z} \frac{\sqrt{1/2}}{\sqrt{2\pi}} e^{-u^2/4} \int_{y=-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1/2)}} e^{-\frac{1}{2}\left(\frac{y-u/2}{\sqrt{1/2}}\right)^2} \; dy \; du \\
&= \int_{u=-\infty}^{z} \frac{1}{\sqrt{2\pi(2)}} e^{-\frac{1}{2}\left(\frac{u}{\sqrt{2}}\right)^2} \; du
\end{aligned}
$$

Problem: Suppose that the waiting time for a bus in the morning is uniformly distributed on [0,8] whereas the waiting time for a bus in the evening is uniformly distributed on [0,10]. Assume that the waiting times are independent.

(a) If you take a bus each morning and evening for a week, what is the total expected waiting time?

(b) What is the variance of total waiting time?

(c) What are the expected value and variance of how much longer you wait in the evening than in the morning on a given day?

Proposition - The Central Limit Theorem (CLT): Let $X_1, \ldots, X_n$ be iid (independent and identically distributed) rvs arising from a distribution with mean μ and variance σ^2. Then as $n \to \infty$,

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \to \text{Normal}(0, 1)$$

Discussion points:

- **the most important (and arguably) most beautiful result in Statistics**
- **weaker versions of the CLT are available**
- **the CLT is important for inference**
- **assuming little, the CLT tells us a lot**
- **try to understand the limits used in the CLT**
- **we use the limiting distribution when the sample size is large ($n \geq 30$)**

We motivate the CLT by considering a sample $X_1, \ldots X_n$ with underlying pmf $p(x)$.

x	1	2	3
$p(x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

(a) Obtain the distribution of $\bar{X}$ when $n = 2$.

(b) Obtain the distribution of $\bar{X}$ when $n = 3$.