Lecture 21

It turns out that there is a connection between the Poisson and Exponential distributions. Recall the Poisson process where N_T is the number of events that occur in the interval $[0, T]$ where $N_T \sim \text{Poisson}(\lambda T)$. Let

$$Y \equiv \textbf{waiting time until the first event}$$

Then the cdf of Y is given by

$$\begin{aligned}
\mathrm{P}(Y \leq y) &= 1 - \mathrm{P}(Y > y) \\
&= 1 - \mathrm{P}(\textbf{zero events in [0,y]}) \\
&= 1 - \mathrm{P}(N_y = 0) \qquad \text{where } N_y \sim \text{Poisson}(\lambda y) \\
&= 1 - (\lambda y)^0 e^{-\lambda y}/0! \\
&= 1 - e^{-\lambda y}
\end{aligned}$$

which implies $Y \sim \text{Exponential}(\lambda)$

Problem: Let X be the distance in metres that a rat moves from its birth site to its first territorial vacancy. Suppose that X has an exponential distribution with $\lambda = 0.01386$.

(a) What is the probability that the distance X is at most 100 metres?

(b) What is the probability that the distance X exceeds the mean distance by more than two standard deviations?

(c) What is the median distance?

Until now, we have studied probabilities corresponding to a single rv X. We now consider joint probability distributions associated with a vector rv $(X_1, \ldots, X_k)$.

Example: a trivariate discrete distribution described by the pmf $p(x, y, z)$

	X=1	**X=2**	**X=3**
Y=1	0.10	0.20	0.00
Y=2	0.00	0.05	0.05

$Z = 5$

	X=1	**X=2**	**X=3**
Y=1	0.00	0.30	0.10
Y=2	0.05	0.05	0.10

$Z = 6$

The *marginal* pmf $\mathrm{p}(x) = \Sigma_{y,z} \mathrm{p}(x, y, z)$

In the continuous setting, we describe distributions via a joint pdf $f(x_1, \ldots, x_k)$ which satisfies

1. $f(x_1, \ldots, x_k) \geq 0 \quad \forall\ x_1, \ldots, x_k$

2. $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_k)\ dx_1 \cdots dx_k \ = \ 1$

To obtain probabilities in the continuous setting,

$\mathrm{P}((X_1, \ldots, X_k) \in A) = \int \cdots \int_A f(x_1, \ldots, x_k)\ dx_1 \cdots dx_k$

Example: A bivariate distribution on (X, Y) is given by $f(x, y) = 2(2x + 3y)/5$ where $0 < x, y < 1$

(a) Calculate $\mathrm{P}(X > 1/2, Y < 1/2)$.

(b) Obtain the marginal pdf of X and verify that it is a pdf.

Recall that we previously discussed the independence of events. The concept of independence can be extended to rv's.

Definition: Random variables are independent if their joint pmfs (pdfs) factor into their marginal pmfs (pdfs).

Example: Consider the bivariate pdf

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left\{-\frac{1}{2}\left(x^2/\sigma_1^2 + y^2/\sigma_2^2\right)\right\} \\ &= \frac{1}{\sqrt{2\pi\sigma_1^2}}\exp\left\{-\frac{1}{2}\left(\frac{x-0}{\sigma_1}\right)^2\right\}\ \frac{1}{\sqrt{2\pi\sigma_2^2}}\exp\left\{-\frac{1}{2}\left(\frac{y-0}{\sigma_2}\right)^2\right\} \end{aligned}$$

Example: Consider the bivariate pmf given by

	X=1	**X=2**
Y=1	0.4	0.2
Y=2	0.1	0.3

(a) Obtain the marginal pmf for X**.**

(b) Obtain the marginal pmf for Y**.**

(c) Are X **and** Y **independent?**

Problem: Two components of a computer have the joint pdf for their lifetimes X and Y in years

$$f(x, y) = xe^{-x(1+y)} \qquad x, y \geq 0$$

(a) What is the probability that the lifetime X of the first component exceeds 3 years?

(b) What are the marginal pdfs of X and Y?

(c) What is the probability that the lifetime of at least one component exceeds 3 years?