Lecture 13

We explore *expectation* in more detail.

Proposition: For a discrete rv X with pmf $p(x)$

$$\mathrm{E}(g_1(x) + \cdots + g_k(x)) = \mathrm{E}(g_1(x)) + \cdots + \mathrm{E}(g_k(x))$$

Definition: The *variance* of a discrete rv X with pmf $p(x)$ is

$$\sigma^2 \equiv \mathrm{V}(X) \equiv E[(X - \mathrm{E}(X))^2]$$

- **we call σ the *standard deviation***
- **σ and σ^2 are measures of spread**
- **contrast sample quantities $(\bar{x}, s)$ with popln quantities (μ, σ)**

Example: Consider the experiment consisting of three flips of a coin. Let $X \equiv$ the number of heads. Obtain $\mathrm{V}(X)$.

Proposition: $\mathrm{V}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

Proposition: $\mathrm{V}(aX + b) = a^2\mathrm{V}(X)$

Example: Let X be the average january temperature in degrees Celsius where $\mathrm{E}(X) = 5$C and $\mathrm{V}(X) = 3(\mathrm{C})^2$. Find the expected value and the variance of Y where Y is the average january temperature in degrees Fahrenheit.

Problem: Calculate σ and $\mathrm{E}(3X + 4X^2)$ corresponding to the rv X with pmf $p(x)$ where

x	4	8	10
$p(x)$	0.2	0.7	0.1

Example: In a game of chance, I bet x dollars. With probability p, I win y dollars. What should x be for this to be a fair game?

Definition: A discrete rv X has a *Binomial* distribution denoted $\mathrm{Bin}(n, \theta)$ if it has pmf

$$p(x) = \begin{cases} \binom{n}{x} \theta^x (1-\theta)^{n-x} & x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

Motivation for the Binomial - the most important discrete distribution:

Consider performing an experiment n times where the probability of success in every trial is θ and the n experiments are independent. We are interested in the probability of x successes.

The probability of getting x successes S and $n-x$ failures F in the specific order

$$SS\cdots S \quad FF\cdots F$$

is

$$\begin{aligned} &\theta\theta\cdots\theta \quad (1-\theta)(1-\theta)\cdots(1-\theta) \\ = \; &\theta^x(1-\theta)^{n-x}. \end{aligned}$$

Therefore

$$P(x \text{ successes}) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$$

for $x = 0, 1, \ldots, n$.

Key points for the Binomial distribution:

1. **the n trials are independent**
2. **same probability of success θ in each trial**

Example: You roll a die 10 times and are interested in obtaining 5's or 6's. What is the probability that x rolls result in either 5's or 6's?

For $X \sim \mathrm{Bin}(n, \theta)$

- $\mathrm{E}(X) = n\theta$
- $\mathrm{V}(X) = n\theta(1-\theta)$

Proof (first result):

$$
\begin{aligned}
\mathrm{E}(X) &= \Sigma_{x=0}^{n}\ x \begin{pmatrix} n \\ x \end{pmatrix} \theta^{x}(1-\theta)^{n-x} \\
&= \Sigma_{x=1}^{n}\ x \tfrac{n!}{x!(n-x)!}\ \theta^{x}(1-\theta)^{n-x} \\
&= n\theta\ \Sigma_{x=1}^{n} \tfrac{(n-1)!}{(x-1)!(n-x)!}\ \theta^{x-1}(1-\theta)^{n-x} \\
&= n\theta\ \Sigma_{y=0}^{n-1} \tfrac{(n-1)!}{y!(n-1-y)!}\ \theta^{y}(1-\theta)^{n-1-y} \\
&= n\theta\ \Sigma_{y=0}^{n-1} \begin{pmatrix} n-1 \\ y \end{pmatrix} \theta^{y}(1-\theta)^{n-1-y} \\
&= n\theta
\end{aligned}
$$